LOCAL AND GLOBAL STRUCTURE OF CONNECTIONS ON NONARCHIMEDEAN CURVES

KIRAN S. KEDLAYA

Abstract. Consider a vector bundle with connection on a p-adic analytic curve in the sense of Berkovich. We collect some improvements and refinements of recent results on the structure of such connections, and on the convergence of local horizontal sections. This builds on work from the author's 2010 book and on subsequent improvements by Baldassarri and Poineau–Pulita.

Introduction

The theory of p-adic ordinary differential equations has been an active part of number theory ever since the pioneering work of Dwork, starting with his p-adic analytic proof of the rationality aspect of the Weil conjectures circa 1960 (predating the development of étale cohomology). The subsequent half-century saw slow but substantial progress on the question of convergence of solutions of p-adic differential equations; in that time, new spheres of application (rigid cohomology, p-adic Hodge theory, numerical computation of zeta functions, p-adic dynamical systems) have attracted additional attention to the area. A broad survey of the theory of p-adic differential equations has been given recently by the author in the book [25].

At about the time that [25] was written, it was observed by Baldassarri [6, 5] that the classical theory of p-adic differential equations could be rearticulated much more clearly using Berkovich's language of analytic geometry over complete nonarchimedean fields. That is because the classical theory is heavily concerned with the convergence of local solutions of p-adic differential equations around certain *generic points*, which appear naturally in Berkovich's framework on an equal footing with rigid analytic point. In this language, one can also naturally treat general p-adic curves, not just subspaces of the affine line, by using semistable models to obtain scaling parameters; Baldassarri

Date: 29 Jan 2013.

The author was supported by NSF (grant DMS-1101343), DARPA (grant HR0011-09-1-0048), MIT (NEC Fund, Cecil and Ida Green professorship), UCSD (Stefan E. Warschawski professorship).

demonstrated this in [5] by establishing continuity of the radius of convergence for a differential module over a semistable p-adic curve.

The radius of convergence function for a differential module over a curve measures only the joint radius of convergence of all local horizontal sections around a point. A finer invariant is the *convergence polygon*, a Newton polygon whose slopes record the extent to which there exist subspaces of the local horizontal sections which converge on larger discs. Building on the results of [25], it has been shown recently by Poineau and Pulita [35, 36] that the convergence polygon is again a continuous function which factors through the retraction onto some finite skeleton (as in the work of Payne [34]). Informally, this means that the convergence of local horizontal sections is controlled by finitely many numerical invariants. A shorter derivation of these results is given by Baldassarri and the present author in [7].

The purpose of this paper is to collect some results about differential modules on nonarchimedean analytic curves over fields of characteristic 0 which refine and extend the aforementioned results as well as some other results from [25]. Here is a partial list of the new results of the present paper.

- We make a finer analysis of refined differential modules over a field of analytic functions than is made in [25]; see §2.3. This leads to results about refined differential modules on open annuli; see §3.7.
- We provide more detailed discussion of the theory of exponents for differential modules on annuli satisfying the Robba condition (existence of horizontal sections over any open disc); see §3.2 and §3.4.
- We generalize the p-adic local monodromy theorem to arbitrary differential modules over an open annulus at one boundary, with no hypotheses on Frobenius structures or p-adic exponents; see §3.8.
- We show that the convergence polygon of a differential module on a curve is constant locally around any point of type 4; see §4.4. This result is used in [7] to strengthen the continuity theorem of [35, 36].
- We show that every curve admits a triangulation such that locally at any interior point, the connection decomposes into a particularly simple form; see §5.4. Such triangulations and decompositions can be used to give a global version of the Christol-Mebkhout index formula; this will appear in a forthcoming paper of Baldassarri.

As in [25], we have made an effort to maintain as much parity as possible between the cases of zero and positive residual characteristic. One unavoidable complication in the latter case is the existence of some pathologies in the theory of regular singularities caused by the existence of p-adic Liouville numbers (numbers which are not integers but which admit extremely good integer approximations). These complications generally emerge when considering cohomology; in this paper, we primarily limit ourselves to statements of a "precohomological" nature, for which one can skirt these complications with some extra work.

Note that while many of the interesting applications of p-adic differential equations involve spaces of dimension greater than 1, in this paper we follow the model of [25] and confine attention to *ordinary* p-adic differential equations. It is of course natural to consider also higher-dimensional spaces; in so doing, one should be able to obtain a unification of some existing work. Such work would include the study of good formal structures for formal flat meromorphic connections [26, 27] in the case of zero residual characteristic and semistable reduction for overconvergent F-isocrystals [22, 23, 24, 28] in the case of positive residual characteristic.

Acknowledgments. Thanks to Francesco Baldassarri for arranging a visit to Padova in September 2012 during which some of this work was completed. Thanks also to Matt Baker, Xander Faber, Andrea Pulita, and Jerôme Poineau for helpful discussions.

1. Preliminaries

We begin with some assorted preliminary definitions and arguments. This also provides an opportunity to set running notation for the whole paper.

Notation 1.0.1. Throughout the paper, let K denote an *analytic field* (a field equipped with a nonarchimedean multiplicative norm $|\cdot|$ with respect to which it is complete) of characteristic 0. Let $\mathfrak{o}_K$, $\mathfrak{m}_K$, and κ_K denote the valuation subring, maximal ideal, and residue field of K, respectively. Let p denote the characteristic of κ_K; put $\omega = 1$ if $p = 0$ and $\omega = p^{-1/(p-1)}$ if $p > 0$. Let $\mathbb{C}$ denote a completed algebraic closure of K.

1.1. **A lemma on linear groups.** In order to analyze the structure of the automorphism groups of certain Tannakian categories, we need a bit of elementary analysis of linear groups in the spirit of André's abstract analysis of filtrations [2]. For the formalism of Tannakian categories, including the Tannaka-Krein duality theorem, see [41].

Lemma 1.1.1. *Let F be a field of characteristic 0. Fix a positive integer n and let $G_0 \subseteq G_1 \subseteq \cdots$ be an increasing sequence of finite subgroups of $\mathrm{GL}_n(F)$ such that G_i is normal in G_j whenever $i \leq j$.*

(a) *The union $G = \bigcup_{i=0}^{\infty} G_i$ contains an abelian normal subgroup H of finite index.*
(b) *There exists an index i such that G/G_i is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$, and in particular is abelian.*

Proof. By Jordan's theorem on finite linear groups [15, Chapter 36], there exists a constant $f(n)$ such that each G_i contains an abelian normal subgroup of index at most $f(n)$. Let S_i be the set of abelian normal subgroups of G_i of index at most $f(n)$. For each $H_j \in S_j$ and each $i \leq j$, the map $G_i/(S_i \cap H_j) \to G_j/H_j$ is injective, so $G_i \cap H_j \in S_i$. We may thus assemble the sets S_i into an inverse system via restriction, and the inverse limit is necessarily nonempty by Tikhonov's theorem. This proves (a).

Given (a), let $\overline{F}$ be an algebraic closure of F. Then H is an abelian torsion group which embeds into $(\overline{F}^*)^n$. This implies that H is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$, as then is any quotient of H. Note also that since the group G/H is finite and is the union of its subgroups $G_i/(G_i \cap H)$, there must exist an index i for which the inclusion $G_i/(G_i \cap H) \to G/H$ is bijective. The group G/G_i is then isomorphic to the abelian group $H/(G_i \cap H)$. This proves (b). $\square$

Proposition 1.1.2. *Let F be a field of characteristic 0. Let V be a finite-dimensional F-vector space. Let G be an algebraic subgroup of $\mathrm{GL}(V)$. Let $\{G^r\}_{r \in \mathbb{R}}$ be a family of normal algebraic subgroups of G. For $r \geq -\infty$, put $G^{r+} = \cup_{s>r} G^s$. Assume also the following conditions.*

(a) *For every $r, s \in \mathbb{R}$ with $r \leq s$, G^s is a normal subgroup of G^r.*
(b) *For every $s \in \mathbb{R}$, there exists $r < s$ such that $G^r = G^s$.*
(c) *There exists $r \in \mathbb{R}$ such that G^r is the trivial group.*
(d) *For every $r \in \mathbb{R}$ for which G^{r+} is finite and all nonnegative integers g, h, the G^{r+}-invariant subspace of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ admits a direct sum decomposition into G-stable subspaces, each of which restricts to an isotypical representation of G^r/G^{r+}.*
(e) *In (d), the isotypical representations of G^r/G^{r+} that occur all have finite image.*
(f) *For all nonnegative integers g, h and every one-dimensional G-stable subspace W of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$, the image of $G^{-\infty+}$ in $\mathrm{GL}(W)$ is finite.*

Then $G^{-\infty+}$ is itself finite.

Proof. Let S be the set of $r \in \mathbb{R}$ for which G^r is finite. By (a), the set S is up-closed. By (b), the set S does not contain its infimum. By (c), the set S is nonempty.

Put $r = \inf S$; by the previous paragraph, $r \notin S$. Suppose by way of contradiction that G^{r+} is infinite. By Lemma 1.1.1, there exists $s_0 > r$ such that G^{r+}/G^{s_0} embeds into a product of finitely many copies of $\mathbb{Q}/\mathbb{Z}$. By Tannaka-Krein duality, we can choose g, h so that $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ contains a G-stable subspace X on which G^{s_0} acts trivially but G^s acts nontrivially for some $s \in (r, s_0)$. By applying (d) finitely many times (with r replaced by varying choices of $s \in (r, s_0)$), we can split X as a direct sum of G-stable summands, each of which is G^{r+}-isotypical. Since G^{r+} is not finite, we can choose a G-stable summand Y of X such that G^{r+} has image in $\mathrm{GL}(Y)$ isomorphic to an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$. Put $W = \wedge^{\dim(Y)} Y$; this space occurs as a G-invariant subspace of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ for some possibly different values of g and h. However, the image of G^{r+} in $\mathrm{GL}(W)$ is again isomorphic to an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$, contradicting (f).

We conclude that G^{r+} is finite. Suppose now that $r \in \mathbb{R}$. By Tannaka-Krein duality, the action of G^r on the direct sum of the G^{r+}-invariant subspaces of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ over all nonnegative integers g, h is a faithful representation of G^r/G^{r+}. However, by (e), the action on each individual summand factors through a finite group; since G^r is algebraic, this implies that G^r is finite. But then $r \in S$, a contradiction. We must thus have $r = -\infty$, which yields the desired result. $\square$

We will apply Proposition 1.1.2 via the following Tannakian interpretation.

Remark 1.1.3. Let F be a field of characteristic 0. Let $\mathcal{C}$ be a Tannakian category equipped with a fibre functor ω to the category of finite-dimensional F-vector spaces. Assign to each nonzero element $V \in \mathcal{C}$ an element $r = r(V) \in \mathbb{R} \cup \{-\infty\}$ depending only on the isomorphism class of V, subject to the following conditions.

(a) For any $V \in \mathcal{C}$, $r(V^\vee) = r(V)$.
(b) For any short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in $\mathcal{C}$, $r(V) = \max\{r(V_1), r(V_2)\}$.
(c) For any $V_1, V_2 \in \mathcal{C}$, $r(V_1 \otimes V_2) \leq \max\{r(V_1), r(V_2)\}$.

For $V \in \mathcal{C}$, let $[V]$ denote the Tannakian subcategory of $\mathcal{C}$ generated by V; note that $r(W) \leq r(V)$ for all $W \in [V]$. Let $G(V) \subseteq \mathrm{GL}(\omega(V))$ denote the automorphism group of the restriction of ω to $[V]$; this is an algebraic group over F, so all of its pro-algebraic subgroups are also algebraic. For $r \in \mathbb{R}$, let $G^r(V)$ be the subgroup of $G(V)$ acting trivially

on $\omega(W)$ for all $W \in [V]$ with $r(W) < r$. For $r \in \mathbb{R} \cup \{-\infty\}$, put $G^{r+}(V) = \cup_{s>r} G^s(V)$; if this group is finite, then it equals the subgroup of $G(V)$ acting trivially on $\omega(W)$ for all $W \in [V]$ with $r(W) \leq r$ (because there exists $s > r$ for which $G^s(V) = G^{r+}(V)$ and hence $r(W) \notin (r, s)$ for all $W \in [V]$).

The groups $G^r(V)$ then satisfy conditions (a),(b),(c) of Proposition 1.1.2. This is evident for (a) and (c). For (b), note that the objects $W \in [V]$ for which $G^s(V)$ acts trivially on $\omega(W)$ form a Tannakian category which is finitely generated (because restricting ω gives a fibre functor whose automorphism group $G(V)/G^s(V)$ is algebraic, not just pro-algebraic).

To enforce conditions (d),(e),(f) of Proposition 1.1.2, it would suffice to have the following additional information about $\mathcal{C}$.

(i) Every $V \in \mathcal{C}$ with $r(V) > -\infty$ admits a direct sum decomposition $V = \bigoplus_i V_i$ in which each summand V_i satisfies $r(V_i^\vee \otimes V_i) < r(V)$. (This implies (d).)

(ii) For every $V \in \mathcal{C}$ with $r(V^\vee \otimes V) < r(V)$, there exists a positive integer n such that $r(V^{\otimes n}) < r(V)$. (Given (i), this implies (e).)

(iii) For every $V \in \mathcal{C}$ with $\dim_F \omega(V) = 1$, there exists a positive integer n such that $r(V^{\otimes n}) = -\infty$. (This implies (f).)

Note also that if in (ii) and (iii) the integer n can always be taken to be a power of a fixed prime p, then the group $G^{-\infty+}(V)$ is then forced to be not only finite but also a p-group.

1.2. **A lemma on local fields.** We introduce an auxiliary calculation concerning local fields in positive characteristic. We use without comment some basic facts about higher ramification of local fields, for which see [25, Chapter 3] for a brief summary or [43] for a complete treatment.

Hypothesis 1.2.1. Throughout §1.2, assume that $p > 0$ and let k be an algebraically closed field of characteristic p.

Definition 1.2.2. Let N be the pro-unipotent pro-algebraic group over k whose k-points are identified with the t-adically continuous k-linear automorphisms ψ of $k((t))$ fixing t modulo t^2. The group N is filtered by the pro-algebraic subgroups

$$N_m = \ker(N \to \mathrm{Aut}(k[\![t]\!]/t^{m+1})) \qquad (m = 1, 2, \dots)$$

for which $N_1 = N$ and each successive quotient N_m/N_{m+1} is isomorphic to the additive group (though not canonically). We will write N_t and $N_{m,t}$ instead of N and N_m when it is necessary to specify the series

variable t in the notation. (The analogous construction with $k = \mathbb{F}_p$ is sometimes called the *Nottingham group*.)

Hypothesis 1.2.3. For the remainder of §1.2, let m be a positive integer, and let Γ_m be a copy of the additive group over k equipped with a homomorphism $\Gamma_m \to N_m$ of pro-algebraic groups over k such that the composition $\Gamma_m \to N_m \to N_m/N_{m+1}$ is surjective and separable.

Example 1.2.4. The key case of Hypothesis 1.2.3 for our intended applications is the one in which $m = 1$ and Γ_m is the group of translations $t^{-1} \mapsto t^{-1} + c$. However, we will need the full generality of Hypothesis 1.2.3 in order to make certain inductive arguments in towers of field extensions.

Lemma 1.2.5. *Let E be a $\mathbb{Z}/p\mathbb{Z}$-extension of $k((t))$ equipped with an extension of the action of Γ_m.*

(a) *The ramification number e of E is a positive integer no greater than m and not divisible by p.*
(b) *Put $m' = (m-e)p + e$. For any k-linear homeomorphism $E \cong k((u))$, the action of Γ_m on E induces a homomorphism $\Gamma_m \to N_{m',u}$ of pro-algebraic groups such that the composition $\Gamma_m \to N_{m',u} \to N_{m',u}/N_{m'+1,u}$ is surjective and separable.*

Proof. Let φ denote the p-power Frobenius endomorphism of $k((t))$. Write E as an Artin-Schreier extension $k((t))[z]/(z^p - z - x)$ with the t-adic valuation of x as large as possible. We then have $x = at^{-e} + \cdots$ for some nonzero $a \in k$, where e is the ramification number of E. In particular, e is a positive integer not divisible by p (it cannot be 0 because k has been assumed to be algebraically closed).

For each $c \in k$, the element $\psi_c \in \Gamma_m$ corresponding to c has the property that $\psi_c(x)$ defines the same Artin-Schreier extension of $k((t))$ as does x, and so the elements x and $\psi_c(x)$ must generate the same $\mathbb{F}_p$-subspace of $\operatorname{coker}(\varphi - 1, k((t)))$. Since x and $\psi_c(x)$ both have the form $at^{-e} + \cdots$ and e is not divisible by p, the images of x and $\psi_c(x)$ in $\operatorname{coker}(\varphi - 1, k((t)))$ must in fact coincide.

Write $\psi_c(t) = t + \sum_{i=m+1}^{\infty} P_i(c)t^i$ for certain polynomials $P_i(T) \in k[T]$. Because $\Gamma_m \to N_m/N_{m+1}$ is separable, P_{m+1} is not a p-th power. Moreover, the map $c \mapsto P_{m+1}(c)$ must be additive in order to come from a group action.

Suppose that $e > m$, and write $x = \sum_{j \geq -e} a_j t^j$ with $a_{-e} = a$. We then have

$$\psi_c(x) - x \equiv \sum_{j=m-e}^{-1} Q_j(c)t^j \pmod{k[\![t]\!]}$$

for certain polynomials $Q_j(T) \in k[T]$, and in particular $Q_{m-e}(T) = -eaP_{m+1}(T)$. Since $\psi_c(x) - x \in \operatorname{coker}(\varphi - 1, k((t)))$, we must have

$$\sum_{i=0}^{\infty} Q_{(m-e)/p^i}(c)^{p^i} = 0 \qquad (c \in k). \tag{1.2.5.1}$$

However, in the sum $\sum_{i=0}^{\infty} Q_{(m-e)/p^i}(T)^{p^i}$, the $i = 0$ term is not a p-th power whereas all of the other terms are. Consequently, (1.2.5.1) asserts that a nonzero polynomial over k vanishes at all $c \in k$, a contradiction. We conclude that $e \leq m$, proving (a).

To prove (b), note that it is sufficient to check the claim for a single k-linear homeomorphism $E \cong k((u))$. We will check the claim with $u = z^r t^s$ for an arbitrary pair of integers r, s satisfying $-re + ps = 1$ (which exist because e is not divisible by p). To begin with, we have $z = Au^{-e} + \cdots, t = Bu^p + \cdots$ for some $A, B \in k$; using the equalities

$$u = z^r t^s, \qquad z^p = at^{-e} + \cdots,$$

we can solve for A and B to obtain

$$z = a^s u^{-e} + \cdots, \qquad t = a^{-r} u^p + \cdots.$$

By (a), we have $m - e \geq 0$. For $c \in k$, we thus have

$$\begin{aligned}
(\psi_c(z) - z)^p - (\psi_c(z) - z) &= \psi_c(x) - x \\
&= (\psi_c - 1)(at^{-e} + \cdots) \\
&= -eaP_{m+1}(c)t^{m-e} + \cdots \in k[\![t]\!].
\end{aligned}$$

In case $e < m$, this implies that $\psi_c(z) = z + eaP_{m+1}(c)t^{m-e} + \cdots$. Since the u-adic valuation of $(\psi_c(z) - z)/z$ is $(m - e)p + e = m'$ while the valuation of $(\psi_c(t) - t)/t$ is the strictly larger value mp, we obtain

$$\begin{aligned}
\psi_c(u) &= \psi_c(z)^r \psi_c(t)^s \\
&= z^r t^s + reaP_{m+1}(c)t^{m-e+s} z^{r-1} + \cdots \\
&= u + reP_{m+1}(c)a^{1-r(m-e)-s} u^{m'+1} + \cdots.
\end{aligned}$$

In case $e = m$, we instead have $\psi_c(z) = z + d + \cdots$ for some $d \in k$ satisfying $d - d^p = eaP_{m+1}(c)$. Computing as before, we obtain

$$\psi_c(u) = u + rda^{-s} u^{m'+1} + \cdots.$$

In both cases, we obtain (b). □

Proposition 1.2.6. *Let E be a finite Galois extension of $k((t))$ equipped with an extension of the action of Γ_m. Then the ramification breaks of $E/k((t))$ in the upper numbering are all less than or equal to m.*

Proof. Note that E is totally ramified because we assumed that k is algebraically closed. Also, by replacing m by a multiple, we may reduce to the case where E is totally wildly ramified. In this case, we induct on the degree of E, the case $E = k((t))$ serving as a trivial base case.

Suppose $E \neq k((t))$. Let e be the least ramification break of E in the upper numbering, and let F_e be the corresponding subfield of E. Since the definition of the ramification filtration is invariant under automorphisms of $k((t))$, we obtain an action of Γ_m on F_e. Moreover, Γ_m acts on $H = \mathrm{Gal}(F_e/k((t)))$ via a discrete quotient, but the additive group has no nontrivial discrete quotients. Consequently, if we pick any $\mathbb{Z}/p\mathbb{Z}$-subextension F of F_e, then Γ_m acts on F.

By Lemma 1.2.5(a), we have $e \leq m$. In addition, if we put $m' = (m-e)p + e$ and choose a homeomorphism $F \cong k((u))$, then by Lemma 1.2.5(b), we obtain a homomorphism $\Gamma_m \to N_{m',u}$ such that the composition $\Gamma_m \to N_{m',u} \to N_{m',u}/N_{m'+1,u}$ is surjective and separable. This last fact allows us to invoke the induction hypothesis, which implies that the ramification breaks of E/F in the upper numbering are all less than or equal to m'. By Herbrand's rule for transferring ramification breaks from a group to a subgroup [43, §IV.3], this in turn implies that the ramification breaks of $E/k((t))$ for the upper numbering are all less than or equal to m, as desired. $\square$

2. Differential modules over complete fields

We next recall some definitions and results from [25] concerning the spectral behavior of differential modules over complete fields. We then make a few additional calculations leading to a finiteness result concerning the Tannakian automorphism group of a differential module.

Convention 2.0.1. For a matrix over a ring equipped with a norm, we will always interpret the norm of the matrix to be the supremum norm over entries of the matrix.

2.1. Differential rings and modules.

We need some general terminology concerning differential rings and modules.

Definition 2.1.1. By a *differential ring*, we will mean a pair (R, d) in which R is a commutative unital ring and d is a derivation on R. By a *differential module* over (R, d), we will mean a pair (M, D) in which M is a finite projective R-module and D is a differential operator on M with respect to d. For example, for each nonnegative integer n, $R^{\oplus n}$ may be viewed as a differential operator by setting $D(r_1, \dots, r_n) = (d(r_1), \dots, d(r_n))$; any differential module isomorphic to one of this

form is said to be *trivial*. We will often omit mention of d and/or D when they may be inferred from context.

Remark 2.1.2. Let M be a differential module over a differential ring R which is freely generated by the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Then the action of D on M can be reconstructed from the matrix N defined by $D(\mathbf{e}_j) = \sum_i N_{ij}\mathbf{e}_i$ (the *matrix of action* of D on the basis). Any other basis $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is uniquely determined by the invertible matrix U over R defined by $\mathbf{e}'_j = \sum_i U_{ij}\mathbf{e}_i$ (the *change-of-basis matrix* from the $\mathbf{e}_i$ to the $\mathbf{e}'_i$); the matrix of action of D on this new basis has the form $U^{-1}NU + U^{-1}d(U)$.

Definition 2.1.3. The differential modules over a given differential ring form a tensor category. For M a differential module, we write $\mathrm{End}(M)$ as shorthand for $M^\vee \otimes M$; there is a natural composition morphism $- \circ - : \mathrm{End}(M) \otimes \mathrm{End}(M) \to \mathrm{End}(M)$.

Definition 2.1.4. Let (M, D) be a differential module of rank n over a differential ring (R, d). A *cyclic vector* for M is an element $\mathbf{v} \in M$ such that $\mathbf{v}, D(\mathbf{v}), \dots, D^{n-1}(\mathbf{v})$ form a basis of M as an R-module.

Lemma 2.1.5 (Cyclic vector theorem)**.** *Let (R, d) be a differential ring such that R is a field of characteristic 0 and d is nonzero. Then every differential module over R admits a cyclic vector.*

Proof. See for instance [25, Theorem 5.4.2]. □

Corollary 2.1.6. *Let (R, d) be a differential ring such that R is a domain of characteristic 0 and d is nonzero. Then every differential module M over (R, d) contains a cyclic vector for $M \otimes_R \mathrm{Frac}(R)$.*

Definition 2.1.7. For (M, D) a differential module, write $H^0(M)$ and $H^1(M)$ for $\ker(D)$ and $\mathrm{coker}(D)$, respectively. Note that $H^1(M)$ may be interpreted as a Yoneda extension group.

2.2. **Differential modules over fields.** We next review some of the theory of differential modules over completed rational function fields (also known as *fields of analytic elements*) as presented in [25, Chapters 9–10].

Hypothesis 2.2.1. Throughout §2.2, choose $\rho > 0$, let F_ρ be the completion of $K(t)$ for the ρ-Gauss norm, and let E be a finite tamely ramified extension of F_ρ. View F_ρ as a differential field for the derivation $d = \frac{d}{dt}$, which extends uniquely to E.

Definition 2.2.2. Let (V, D) be a differential module over E. For V nonzero, let $IR(V)$ denote the *intrinsic radius* of V in the sense of [25,

Definition 9.4.7]. That is, $\omega/(\rho IR(V))$ equals the spectral radius of D as a K-linear endomorphism of V for any E-Banach norm on V. The following properties are easily derived (see [25, Lemma 6.2.8]).

(a) For any V, $IR(V^\vee) = IR(V)$.
(b) For any short exact sequence $0 \to V_1 \to V \to V_2 \to 0$, $IR(V) = \min\{IR(V_1), IR(V_2)\}$.
(c) For any V_1, V_2, $IR(V_1 \otimes V_2) \geq \min\{IR(V_1), IR(V_2)\}$, with equality if $IR(V_1) \neq IR(V_2)$.

As in [25, Definition 9.8.1], the multiset of *intrinsic subsidiary radii* of V is constructed as follows: for each Jordan-Hölder constituent W of V, include $IR(W)$ with multiplicity $\dim_{F_\rho}(W)$. This multiset is invariant under arbitrary extensions of the constant field and under finite tamely ramified extensions of E [25, Proposition 10.6.6], and its maximum element equals $IR(V)$.

Let $s_1 \leq \cdots \leq s_n$ be the intrinsic subsidiary radii of V. The *spectral polygon* of V, denoted $P(V)$, is then defined to be the convex polygonal curve starting at $(-n, 0)$ and consisting of segments of width 1 and slopes $\log s_1, \dots, \log s_n$ in that order.

Definition 2.2.3. Let V be a nonzero differential module over E. We say V is *pure* if its intrinsic subsidiary radii are all equal. We say that V is *refined* if $IR(\mathrm{End}(V)) > IR(V)$; this condition implies that $IR(V) < 1$, and also that V is pure (using [25, Lemma 9.3.4]). Consequently, this definition of refinedness agrees with that of [25, Definition 6.2.12].

We say that two refined differential modules V_1, V_2 over E are *equivalent* if $IR(V_1) = IR(V_2) < IR(V_1^\vee \otimes V_2)$. As the terminology suggests, this is an equivalence relation [25, Lemma 6.2.14].

Lemma 2.2.4. *Let V_1, V_2 be nonzero differential modules over F_ρ such that $IR(V_1), IR(V_2) < IR(V_1^\vee \otimes V_2)$. Then $IR(V_1) = IR(V_2)$ and*

$$IR(\mathrm{End}(V_1)), IR(\mathrm{End}(V_2)) \geq IR(V_1^\vee \otimes V_2);$$

consequently, V_1 and V_2 are both refined of the same intrinsic radius.

Proof. The first claim holds because V_2 is a direct summand of $V_1 \otimes (V_1^\vee \otimes V_2)$ and V_1 is a direct summand of $V_2 \otimes (V_1^\vee \otimes V_2)^\vee$. The second claim holds because $V_1^\vee \otimes V_1$ is a direct summand of $V_1^\vee \otimes V_1 \otimes V_2^\vee \otimes V_2 \cong (V_1^\vee \otimes V_2)^\vee \otimes (V_1^\vee \otimes V_2)$. □

Remark 2.2.5. The intrinsic subsidiary radii of V behave for many purposes like the reciprocal norms of the eigenvalues of some linear transformation associated to V. In this model, a refined differential module (resp. two equivalent refined modules) over E corresponds to a

linear transformation (resp. two linear transformations) whose eigenvalues all have a single image in the graded ring associated to an algebraic closure of F_ρ.

For radii in the range $(0, \omega)$ (called the *visible range* in [25]), this intuition can be made precise using cyclic vectors; see Proposition 2.2.6 below. When $p > 0$, one must use pullback and pushforward along Frobenius to access radii in the range $[\omega, 1)$, as described in [25, Chapter 10]. We will see these techniques in action in §2.3.

Proposition 2.2.6 (Christol-Dwork)**.** *Let V be a differential module over E of rank n, let $\mathbf{v}$ be a cyclic vector of V, and write $D^n(\mathbf{v}) = a_0\mathbf{v} + \cdots + a_{n-1}D^{n-1}(\mathbf{v})$ with $a_0, \dots, a_{n-1} \in E$. Then the multiset of slopes of the spectral polygon of V less than $\log \omega$ consists of $\log \omega - \log \rho + s$ for s running over the multiset of slopes of the Newton polygon of the polynomial $T^n - a_{n-1}T^{n-1} - \cdots - a_0 \in E[T]$ less than $\log \rho$.*

Proof. See [25, Corollary 6.5.4]. □

Corollary 2.2.7. *For any $s < \omega$ and any positive integers n_1, n_2, m, there exists $\delta \in (s, \omega)$ for which the following statements hold. For $i = 1, 2$, let V_i be a differential module over E of rank n_i which is pure of intrinsic radius s. Let $\mathbf{v}_i$ be a cyclic vector of V_i, write $D^{n_i}(\mathbf{v}_i) = a_{0,i}\mathbf{v}_i + \cdots + a_{n_i-1,i}D^{n_i-1}(\mathbf{v}_i)$ with $a_{0,i}, \dots, a_{n_i-1,i} \in E$, and define the polynomial $P_i(T) = T^{n_i} - a_{n_i-1,i}T^{n_i-1} - \cdots - a_{0,i} \in E[T]$.*

- (a) *Let $P(T) \in E[T]$ be the monic polynomial of degree n_1n_2 with roots $\alpha_2 - \alpha_1$ where α_i runs over the roots of P_i. Then the multiset of slopes of the spectral polygon of $V_1^\vee \otimes V_2$ less than $\log \delta$ consists of $\log \omega - \log \rho + c$ for c running over the multiset of slopes of the Newton polygon of $P(T)$ less than $\log \delta - \log \omega + \log \rho$.*
- (b) *Let $Q(T) \in E[T]$ be the monic polynomial of degree n_1^m with roots $\alpha_1 + \cdots + \alpha_m$ where α_i runs over the roots of P_1. Then the multiset of slopes of the spectral polygon of $V_1^{\otimes m}$ less than $\log \omega$ consists of $\log \omega - \log \rho + s$ for c running over the multiset of slopes of the Newton polygon of $Q(T)$ less than $\log \delta - \log \omega + \log \rho$.*

Proof. We describe only (a) in detail, as the proof of (b) is similar. Equip V_i with a norm as in the proof of [25, Theorem 6.5.3]; by enlarging K if necessary, we may ensure that this norm is the supremum norm defined by a basis. Equip $V_1^\vee$ with the dual basis, then equip $V_1^\vee \otimes V_2$ with the product basis and the resulting supremum norm. The claim then follows by applying [25, Theorem 6.7.4] . □

Definition 2.2.8. Let V be a differential module over E. A *spectral decomposition* of V is a direct sum decomposition $V = \bigoplus_{s \in (0,1]} V_s$ such that the intrinsic subsidiary radii of V_s are all equal to s. A *refined decomposition* of V is a direct sum decomposition of V refining a spectral decomposition in which V_1 remains whole, but each V_s with $s < 1$ is split into inequivalent refined summands.

Proposition 2.2.9. *Let V be a differential module over E.*

(a) *There exists a unique spectral decomposition of V.*
(b) *A refined decomposition of V is unique if it exists. Moreover, there exists a finite tamely ramified extension E' of E such that $V \otimes_E E'$ admits a refined decomposition.*

Proof. By restriction of scalars, we may reduce to the case $E = F_\rho$. In this case, see [25, Theorem 10.6.2, Theorem 10.6.7] for (a) and (b), respectively. $\square$

Corollary 2.2.10. *Let V be a differential module over E such that $IR(V) < 1$.*

(a) *If V is indecomposable, then V is pure.*
(b) *If $V \otimes_E E'$ is indecomposable for every finite tamely ramified extension E' of E, then V is refined.*

In case $p = 0$, one can state an even stronger version of Corollary 2.2.10, closely related to the classical Turrittin-Levelt-Hukuhara decomposition theorem for formal meromorphic connections (see for instance [25, Chapter 7]).

Proposition 2.2.11. *Assume $p = 0$. Let V be a differential module over E such that $IR(V) < 1$. If $V \otimes_E E'$ is indecomposable for every finite tamely ramified extension E' of E, then there exists a differential module W over E of dimension 1 such that $IR(W^\vee \otimes V) = 1$.*

Proof. Put $n = \dim_E(V)$. Let $\mathbf{v}$ be a generator of $\wedge^n V$ and define $f \in E$ by the formula $D(\mathbf{v}) = f\mathbf{v}$. Let W be the differential module of dimension 1 over E on the generator $\mathbf{w}$ for which $D(\mathbf{w}) = (f/n)\mathbf{w}$; then $W^{\otimes n} \cong \wedge^n V$, so $\wedge^n(W^\vee \otimes V)$ is trivial. If $IR(W^\vee \otimes V) < 1$, then by Corollary 2.2.10, $W^\vee \otimes V$ would be refined; however, [25, Proposition 6.8.4] would then imply that $IR(W^\vee \otimes V) = IR(\wedge^n(W^\vee \otimes V)) = 1$, a contradiction. Hence $IR(W^\vee \otimes V) = 1$ as desired. $\square$

2.3. **More on refined modules.** Proposition 2.2.11 gives a fairly precise description of the indecomposable differential modules over finite tamely ramified extensions of F_ρ in case $p = 0$. We next turn to the situation where $p > 0$, in which case things are more complicated.

Hypothesis 2.3.1. Throughout §2.3, retain Hypothesis 2.2.1, but assume in addition that $p > 0$. Let μ_p denote the group of p-th roots of unity in some algebraic closure of K.

We recall the basic formalism of Frobenius pullback and pushforward, as in [25, Chapter 10].

Definition 2.3.2. For each $\zeta \in \mu_p$, the K-linear substitution $t \mapsto \zeta t$ induces a continuous automorphism ζ^* of $E(\mu_p)$. Let E' be the fixed subfield of $E(\mu_p)$ under the group generated by $\mathrm{Gal}(E(\mu_p)/E)$ and the automorphisms ζ^* for $\zeta \in \mu_p$; we may then view E' as a differential field for the derivation $d = \frac{d}{dt^p}$, and thus define the intrinsic radius of a nonzero differential module (V', D') over E' so that $\rho^p/(\omega IR(V'))$ equals the spectral radius of D'.

For $m = 0, \dots, p-1$, let (W_m, D') denote the differential module over E' on the single generator $\mathbf{v}$ given by $D'(\mathbf{v}) = (m/p)t^{-p}\mathbf{v}$. By Proposition 2.2.6, $IR(W_m) = \omega^p$ for $m \neq 0$ (see also [25, Definition 10.3.3]).

For (V', D') a differential module over E', define the differential module $\varphi^* V'$ over E to have underlying module $V' \otimes_{E'} E$ and derivation given by $D = D' \otimes pt^{p-1}$.

For (V, D) a differential module over E, define the differential module $\varphi_* V$ over E' to have underlying module V and derivation given by $D' = p^{-1}t^{1-p}D$.

Lemma 2.3.3. *For any nonzero differential module V' over E', $IR(\varphi^* V') \geq \min\{IR(V')^{1/p}, pIR(V')\}$.*

Proof. See [25, Lemma 10.3.2] □

Proposition 2.3.4. *Let V be a nonzero differential module over E such that $IR(V) > \omega$. Then there exists a unique differential module V' over E' (called the* Frobenius antecedent *of V) such that $IR(V') > \omega^p$ and $\varphi^* V' \cong V$; moreover, this module satisfies $IR(V') = IR(V)^p$.*

Proof. See [25, Theorem 10.4.2]. □

Proposition 2.3.5. *Let V be a differential module over E with intrinsic subsidiary radii $s_1, \dots, s_n$. Then the intrinsic subsidiary radii of $\varphi_* V$ (called the* Frobenius descendant *of V) comprise the multiset*

$$\bigcup_{i=1}^{n} \begin{cases} \{s_i^p\} \cup \{\omega^p \text{ } (p-1 \text{ times})\} & \text{if } s_i > \omega \\ \{p^{-1}s_i \text{ } (p \text{ times})\} & \text{if } s_i \leq \omega. \end{cases}$$

Proof. See [25, Theorem 10.5.1]. □

Lemma 2.3.6. (a) *For V a differential module over E, there are canonical isomorphisms*
$$\iota_m : (\varphi_* V) \otimes W_m \cong \varphi_* V \qquad (m = 0, \dots, p-1).$$
(b) *For V a differential module over E, a submodule U of $\varphi_* V$ has the form $\varphi_* X$ for some differential submodule X of V if and only if $\iota_m(U \otimes W_m) = U$ for $m = 0, \dots, p-1$.*
(c) *For V' a differential module over E', there is a canonical isomorphism*
$$\varphi_* \varphi^* V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m).$$

Proof. See [25, Lemma 10.3.6(a,b,c)]. □

Lemma 2.3.7. *Let V' be an indecomposable differential module over E' of intrinsic radius ω^p such that $IR(\varphi^* V') > \omega$. Then there exists a unique $m \in \{0, \dots, p-1\}$ such that $IR(V' \otimes W_m) > \omega^p$.*

Proof. By Proposition 2.3.5, at least one of the intrinsic subsidiary radii of $\varphi_* \varphi^* V'$ is greater than ω^p. By Lemma 2.3.6(c), we have $\varphi_* \varphi^* V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m)$, so for some m, at least one of the intrinsic subsidiary radii of $V' \otimes W_m$ is greater than ω^p. Since $V' \otimes W_m$ is indecomposable, this implies $IR(V' \otimes W_m) > \omega^p$ by Corollary 2.2.10. This proves the existence of m; uniqueness holds because $IR(W_m) = \omega^p$ for $m \neq 0$. □

Corollary 2.3.8. *Let V' be a nonzero differential module over E' of intrinsic radius ω^p such that $IR(\varphi^* V') > \omega$. Then there exists a unique direct sum decomposition $V' = \bigoplus_{m=0}^{p-1} V'_m$ such that $IR(V'_m \otimes W_m) > \omega^p$ for $m = 0, \dots, p-1$.*

Lemma 2.3.9. *Let V'_1, V'_2 be nonzero differential modules over E' of intrinsic radius ω^p such that V'_1 is refined, V'_2 is indecomposable, and $IR(\varphi^*((V'_1)^\vee \otimes V'_2)) > \omega$. Then there exists a unique $m \in \{0, \dots, p-1\}$ such that $IR((V'_1)^\vee \otimes V'_2 \otimes W_m) > \omega^p$.*

Proof. By Corollary 2.3.8, we have a decomposition $(V'_1)^\vee \otimes V'_2 = \bigoplus_{m=0}^{p-1} X_m$ such that $IR(X_m \otimes W_m) > \omega^p$ for $m = 0, \dots, p-1$. Contracting with V'_1 produces an inclusion $V'_2 \to \bigoplus_{m=0}^{p-1} (V'_1 \otimes X_m)$; since V'_2 is indecomposable, we have $V'_2 \subseteq V'_1 \otimes X_m$ for some m. Therefore
$$\begin{aligned} IR((V'_1)^\vee \otimes V'_2 \otimes W_m) &\geq IR((V'_1)^\vee \otimes V'_1 \otimes X_m \otimes W_m) \\ &\geq \min\{IR((V'_1)^\vee \otimes V'_1), IR(X_m \otimes W_m)\} \\ &> \omega^p. \end{aligned}$$
Again, m is unique because $IR(W_m) = \omega^p$ for $m \neq 0$. □

Remark 2.3.10. Let V be a refined differential module over E of intrinsic radius ω such that $\varphi_* V$ admits a refined decomposition $\bigoplus_i X_i$. By Lemma 2.3.6(a), there are canonical isomorphisms $\psi_m : (\varphi_* V) \otimes W_m \cong \varphi_* V$ for $m = 0, \dots, p-1$; we may view these as an action of $\mathbb{Z}/p\mathbb{Z}$ on $\varphi_* V$, which induces an action on the collection of the X_i. Since $IR(W_m) = \omega^p$ for $m \neq 0$, any two distinct X_i in the same orbit are refined and pairwise inequivalent. By Lemma 2.3.6(b), the X_i in a single orbit constitute the pushforward of a direct summand of V.

Lemma 2.3.11. *Let V be a refined differential module over E of intrinsic radius $s \geq \omega$. Then for some finite unramified extension E'_1 of E', there exists a refined differential module V' over E'_1 of intrinsic radius s^p such that $\varphi^* V' \cong V \otimes_{E'} E'_1$.*

Proof. In case $s > \omega$, we may take V' to be the Frobenius antecedent of V (Proposition 2.3.4); we thus assume $s = \omega$ hereafter. Suppose first that V is indecomposable. By Proposition 2.3.5, the intrinsic subsidiary radii of $\varphi_* V$ are all equal to ω^p. We may thus apply Proposition 2.2.9 to produce a finite Galois tamely ramified extension E'_1 of E' such that $\varphi_* V \otimes_{E'} E'_1$ admits a refined decomposition $\bigoplus_i X_i$.

Define an action of $\mathbb{Z}/p\mathbb{Z}$ on the collection of the X_i as in Remark 2.3.10. Since we assumed V is indecomposable, it follows that the X_i form a single orbit under $\mathbb{Z}/p\mathbb{Z}$.

The group $G = \mathrm{Gal}(E'_1/E')$ also acts on the set of the X_i; since W_m is defined over E', this action defines a homomorphism $G \to \mathbb{Z}/p\mathbb{Z}$. We may replace E'_1 with the fixed field of the kernel of this homomorphism; this field has tame degree over E' dividing p and so must be unramified.

Let V' be any of the X_i. By adjunction, the inclusion $V' \to \varphi_* V \otimes_{E'} E'_1$ corresponds to a map $\varphi^* V' \to V \otimes_{E'} E'_1$. Pushing forward gives a new map $\varphi_* \varphi^* V' \to \varphi_* V \otimes_{E'} E'_1$; using Lemma 2.3.6 again, we may rewrite the left side as $\bigoplus_{m=0}^{p-1} (V' \otimes W_m)$ and match up the actions of $\mathbb{Z}/p\mathbb{Z}$. This shows that each composition $V' \otimes W_m \to \varphi_* \varphi^* V' \to \varphi_* V \otimes_{E'} E'_1$ is injective; since distinct terms $V' \otimes W_m$ cannot have isomorphic submodules (because they are refined and inequivalent), the map $\varphi_* \varphi^* V' \to \varphi_* V \otimes_{E'} E'_1$ must be injective. By counting dimensions, this map is also surjective; hence $\varphi^* V' \to V \otimes_{E'} E'_1$ is also bijective. This proves the claim in case V is indecomposable.

For general V (still assuming $s = \omega$), we may split V as a direct sum $\bigoplus_{i=0}^r V_i$ of indecomposable summands. For some E'_1, by the previous arguments there exist differential modules V'_i over E'_1 which are refined of intrinsic radius s^p such that $\varphi^* V'_i \cong V_i \otimes_{E'} E'_1$. By Lemma 2.3.9, for each i we can find $m_i \in \{0, \dots, p-1\}$ such that $IR((V'_0)^\vee \otimes V'_i \otimes W_{m_i}) > \omega^p$. We may thus take $V' = \bigoplus_{i=0}^r V'_i \otimes W_{m_i}$. $\square$

Lemma 2.3.12. *Let V be a pure differential module over E of intrinsic radius $s \geq \omega$ such that $\varphi_* V$ admits a refined decomposition. Group summands in this decomposition according to their $\mathbb{Z}/p\mathbb{Z}$-orbit as per Remark 2.3.10. Then the resulting decomposition descends to a refined decomposition of V.*

Proof. We may use Proposition 2.3.4 to check the claim when $s > \omega$, so we may assume $s = \omega$ hereafter. The claim may be checked after enlarging E, so by Proposition 2.2.9 we may ensure that V itself admits a refined decomposition $\bigoplus_i V_i$. After enlarging E again, by Lemma 2.3.11 we may ensure that each V_i can be written as $\varphi^* V_i'$ for some refined differential module V_i' over E'. By Lemma 2.3.6(c), we then have $\varphi_* V_i \cong \bigoplus_{m=0}^{p-1} (V_i' \otimes W_m)$. For i, j distinct and $m \in \{0, \dots, p-1\}$, we cannot have $IR((V_i')^\vee \otimes V_j' \otimes W_m) > \omega^p$ or else Proposition 2.3.5 would imply $IR(V_i^\vee \otimes V_j) > \omega$. It follows that the $V_i' \otimes W_m$ are refined and pairwise inequivalent, so they form the refined decomposition of $\varphi_* V$. This proves the claim. □

Proposition 2.3.13. *Let V be a refined differential module over E. Then $IR(V^{\otimes p}) > IR(V)$.*

Proof. It is sufficient to prove that for each nonnegative integer m, the claim holds when $IR(V) < \omega^{p^{-m}}$. For $m = 0$, this follows by Corollary 2.2.7(a,b) with the parameter m in (b) taken to be p. Given this assertion for some m, we may check it for $m+1$ by forming a module V' as in Lemma 2.3.11, applying the known case to deduce that $IR((V')^{\otimes p}) > IR(V') = IR(V)^p$, then observing that $\varphi^*((V')^{\otimes p}) = V^{\otimes p}$ and invoking Lemma 2.3.3 to deduce that $IR(V^{\otimes p}) > IR(V)$. □

When V has dimension 1, we can prove an even stronger assertion.

Lemma 2.3.14. *Let V be a differential module over E of dimension 1. Then*
$$\min\{\omega, IR(V^{\otimes p})\} = \min\{\omega, pIR(V)\}.$$

Proof. This is immediate from Proposition 2.2.6. □

Lemma 2.3.15. *Let V be a differential module over E of dimension 1 such that $\omega^p \leq IR(V) \leq \omega$. Then $IR(V^{\otimes p}) \geq IR(V)^{1/p}$.*

Proof. By enlarging K and rescaling, we may reduce to the case $\rho = 1$. Put $d = \frac{d}{dt}$ and $s = IR(V)$. Choose a generator $\mathbf{v}$ of V and write $D(\mathbf{v}) = n\mathbf{v}$ with $n \in E$. By Proposition 2.2.6, $|n| = \omega/s$. The differential module $V^{\otimes p}$ is generated by $\mathbf{v}^{\otimes p}$ and $D(\mathbf{v}^{\otimes p}) = pn\mathbf{v}^{\otimes p}$. Since $|d|_E = 1$ and $|pn| = p^{-1}\omega/s = \omega^p/s \leq 1$, for any $a \in E$ we have $D^p(a\mathbf{v}^{\otimes p}) = b\mathbf{v}^{\otimes p}$ for some $b \in E$ with $|b - d^p(a)| \leq p^{-1}(\omega/s)\,|a|$. Since

$|d^p|_E = p^{-1} \leq p^{-1}\omega/s$, we conclude that the operator norm of D^p on V is at most $p^{-1}\omega/s = \omega^p/s$, so the spectral norm of D on V is at most $\omega/s^{1/p}$. This implies the desired inequality. $\square$

Proposition 2.3.16. *Let V be a differential module over E of dimension 1 such that $IR(V) < 1$. Then $IR(V^{\otimes p}) \geq \min\{IR(V)^{1/p}, pIR(V)\}$.*

Proof. The claim is trivial if $IR(V) = 1$, so we may assume $IR(V) < 1$. If $IR(V) \leq \omega^p$, then $\min\{IR(V)^{1/p}, pIR(V)\} = pIR(V)$, and in this case the claim follows from Lemma 2.3.14. To complete the proof, it suffices to check the claim in case $\omega^{p^{-h+1}} \leq IR(V) < \omega^{p^{-h}}$ for some nonnegative integer h. We prove this by induction on h, with the base case $h = 0$ following from Lemma 2.3.15. Given the claim for $h-1$, we may deduce the claim for h by forming V' as in Lemma 2.3.11 (after enlarging E if necessary), applying the induction hypothesis to V', and then applying Lemma 2.3.3. $\square$

We are now ready to deduce a finiteness theorem for Tannakian automorphism groups.

Theorem 2.3.17. *Let V be a differential module over E. Let $[V]$ be the Tannakian category of differential modules over E generated by V. Let ω be the fibre functor on $[V]$ which extracts underlying E-vector spaces. Let G be the automorphism group of ω. For $s < 1$, let G^s be the subgroup of G acting trivially on $\omega(W)$ for every $W \in [V]$ with $IR(W) > s$. Then G^s is a finite p-group.*

Proof. Instead of working with differential modules over E, we work with the direct limit of the categories of differential modules over all finite tamely ramified extensions of E; this does not change the groups G^s except for a base extension. In this larger category, we may apply Proposition 1.1.2 using Remark 1.1.3: conditions (i), (ii), (iii) of the remark may be verified using Proposition 2.2.9(b), Proposition 2.3.13, Proposition 2.3.16, respectively. $\square$

Remark 2.3.18. The group $\bigcup_{s<1} G^s$ need not be finite in general. For example, if V is free on one generator $\mathbf{v}$ and $D(\mathbf{v}) = \lambda t^{-1}\mathbf{v}$ for some $\lambda \in K \setminus \mathbb{Q}_p$, then $IR(V^{\otimes n}) < 1$ for all positive integers n [25, Example 9.5.2] and so $\bigcup_{s<1} G^s \cong \mathbb{Q}_p/\mathbb{Z}_p$.

In order to obtain finiteness for some class of differential modules, one must impose additional hypotheses to ensure that when V is of dimension 1, there exists a nonnegative integer m for which $IR(V^{\otimes p^m}) = 1$. For an example of such hypotheses, see Theorem 3.8.12.

Remark 2.3.19. If we assume $p = 0$ but otherwise set notation as in Theorem 2.3.17, then the group $\bigcup_{s<1} G^s$ becomes a torus, as one may deduce easily from Proposition 2.2.11.

3. Differential modules over discs and annuli

We next continue in the vein of [25], treating differential modules on discs and annuli. In this section, we maintain continuity with [25] by phrasing everything in the language of modules over rings of convergent power series. Starting in §4, we will switch to the language of Berkovich spaces in order to articulate more precise and general results.

3.1. **Rings of convergent power series.**

We first introduce the relevant rings of convergent power series on a disc or annulus, modifying the notation somewhat from that used in [25, Chapter 8].

Definition 3.1.1. For $\rho \in [0, +\infty)$, let $|\cdot|_\rho$ denote the ρ-Gauss seminorm on $K[t]$, defined by the formula $|\sum_n c_n t^n|_\rho = \max\{|c_n| \rho^n\}$. For I a subinterval of $[0, +\infty)$, let R_I denote the Fréchet completion of $K[t]$ (if $0 \in I$) or $K[t, t^{-1}]$ (if $0 \notin I$) for the seminorms $|\cdot|_\rho$ for $\rho \in I$. View R_I as a differential ring for the derivation $\frac{d}{dt}$. We will occasionally write $R_{I,K}$ instead of R_I when it is necessary to specify K.

Remark 3.1.2. Let us briefly recall how the rings R_I appear in the notation of [25].

- If $I = [0, \beta]$, then R_I appears as $K\langle t/\beta \rangle$, the ring of analytic functions on the closed disc $|t| \leq \beta$.
- If $I = [0, \beta)$, then R_I appears as $K\{t/\beta\}$, the ring of analytic functions on the open disc $|t| < \beta$.
- If $I = [\alpha, \beta]$ with $\alpha > 0$, then R_I appears as $K\langle \alpha/t, t/\beta \rangle$, the ring of analytic functions on the closed annulus $\alpha \leq |t| \leq \beta$.
- If $I = (\alpha, \beta)$ with $\alpha > 0$, then R_I appears as $K\{\alpha/t, t/\beta\}$, the ring of analytic functions on the open annulus $\alpha < |t| < \beta$.

Remark 3.1.3. Suppose that I is a closed interval. Then R_I is an affinoid algebra for the norm $|\cdot|_I = \sup\{|\cdot|_\rho : \rho \in I\}$. By the log-convexity of $|\cdot|_\rho$ [25, Proposition 8.2.3] (see also Lemma 3.1.5), one has $|\cdot|_{[0,\beta]} = |\cdot|_\beta$ and $|\cdot|_{[\alpha,\beta]} = \max\{|\cdot|_\alpha, |\cdot|_\beta\}$ for $\alpha > 0$. In addition, the ring R_I is a principal ideal domain [25, Proposition 8.3.2], so the underlying module of any differential module over R_I is automatically finite free.

Now let I be arbitrary. In this case, R_I is a *Fréchet-Stein algebra* in the sense of [42, Section 3]; this means that every coherent sheaf on the associated analytic space is generated by its module of global sections. Moreover, any coherent locally free sheaf of rank n is uniformly finitely

generated (because exactly n generators are needed over any closed disc or annulus), and so corresponds to a finite projective module over R_I by [31, Proposition 2.1.15].

Definition 3.1.4. For $x \in \mathbb{R}$, let $\langle x \rangle$ denote the distance from x to the nearest integer, that is, $\langle x \rangle = \min\{x - \lfloor x \rfloor, -x - \lfloor -x \rfloor\}$. We will frequently use the fact that for m a positive integer, $m\langle x/m \rangle$ is the distance from x to the nearest multiple of m.

Lemma 3.1.5. *Choose $\eta > 1$ and $\alpha, \alpha', \beta, \beta' \in [0, +\infty)$ such that*

$$\alpha' < \beta', \qquad \alpha' = \alpha\eta, \qquad \beta' = \beta/\eta.$$

Choose a positive integer m, an element $h \in \mathbb{Z}$, and an element $f \in R_{[\alpha,\beta]}$ whose terms all have exponents congruent to h modulo m.

(a) *Put $h' = m\langle h/m \rangle$. Then*

$$|f|_{[\alpha',\beta']} \leq \eta^{-h'} |f|_{[\alpha,\beta]}.$$

(b) *Assume $h = 0$. Let f_0 be the constant coefficient of f. Then*

$$|f - f_0|_{[\alpha',\beta']} \leq \eta^{-m} |f|_{[\alpha,\beta]}.$$

Proof. Both assertions reduce at once to the case $f = t^n$ for some $n \in h + m\mathbb{Z}$, for which the claim is evident. □

We will also need the construction of rings of analytic elements.

Definition 3.1.6. Let J be the closure of I. Let R_I^{an} be the Fréchet completion of the ring of rational functions in $K(t)$ with no poles in the region $|t| \in I$ for the norms $|\cdot|_\rho$ for $\rho \in J$. This is called the ring of *analytic elements* in the region $|t| \in I$; it is a principal ideal domain [25, Proposition 8.5.2].

- If I is closed, then $R_I^{\mathrm{an}} = R_I$.
- If $I = [0, \beta)$, then R_I^{an} appears in [25] as $K[\![t/\beta]\!]_{\mathrm{an}}$.
- If $I = (\alpha, \beta)$ with $\alpha > 0$, then R_I appears in [25] as $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$.

3.2. **The Robba condition.** We now introduce a special class of differential modules over annuli; this class is closely related to the class of *regular* meromorphic differential modules on a Riemann surface.

Hypothesis 3.2.1. Throughout §3.2, let I be an open subinterval of $[0, +\infty)$ and let M be a differential module of rank n over R_I for the derivation $t\frac{d}{dt}$. For $\rho \in I \setminus \{0\}$, put $M_\rho = M \otimes_{R_I} F_\rho$; for J a closed subinterval of I of positive length, put $M_J = M \otimes_{R_I} R_J$.

Definition 3.2.2. We say that M satisfies the *Robba condition* if $IR(M_\rho) = 1$ for all $\rho \in I - \{0\}$. In this case, we may define an action of the multiplicative group $1 + \mathfrak{m}_K$ on M by the formula

$$\lambda(\mathbf{v}) = \sum_{i=0}^{\infty} \frac{(\lambda - 1)^i}{i!} D^i(\mathbf{v}) \qquad (\lambda \in 1 + \mathfrak{m}_K, \mathbf{v} \in M),$$

since the Taylor series on the right is guaranteed to converge. We may also interpret the action of $\lambda \in 1 + \mathfrak{m}_K$ as an isomorphism $\lambda^*(M) \cong M$, where λ^* is the pullback along the substitution $t \mapsto \lambda t$.

Example 3.2.3. For $\lambda \in K$, let M_λ denote the differential module over R_I on a single generator $\mathbf{v}$ satisfying $D(\mathbf{v}) = \lambda\, dv$. If $p = 0$, then M_λ satisfies the Robba condition whenever $|\lambda| \leq 1$, and is trivial if and only if $\lambda \in \mathbb{Z}$. By contrast, if $p > 0$, then M_λ satisfies the Robba condition if and only if $\lambda \in \mathbb{Z}_p$ [25, Example 9.5.2], and is again trivial if and only if $\lambda \in \mathbb{Z}$ [25, Proposition 9.5.3].

Definition 3.2.4. For A a finite multisubset of $\mathfrak{o}_{K^{\mathrm{alg}}}$, we say A is *prepared* if no two elements a_1, a_2 of A have the property that $|a_1 - a_2 - m| < 1$ for some nonzero integer m. For A, B two finite multisubsets of $\mathfrak{o}_{K^{\mathrm{alg}}}$ of the same cardinality n, we say A and B are *equivalent* if there exist orderings $a_1, \dots, a_n$ and $b_1, \dots, b_n$ of A and B, respectively, such that $a_i - b_i \in \mathbb{Z}$ for $i = 1, \dots, n$; this indeed defines an equivalence relation.

Definition 3.2.5. We say M is of *cyclic type* if $\mathrm{End}(M)$ satisfies the Robba condition. For example, if there exists a differential module N over R_I of positive rank such that $N^\vee \otimes M$ satisfies the Robba condition, then M is of cyclic type by Lemma 2.2.4. Note that the tensor product of modules of cyclic type is again of cyclic type.

Lemma 3.2.6. *Suppose that M is of cyclic type. For each $\lambda \in 1 + \mathfrak{m}_K$, view the Taylor isomorphism $T_\lambda : \lambda^*(\mathrm{End}(M)) \cong \mathrm{End}(M)$ as a horizontal element of*

$$\begin{aligned}
\lambda^*(M^\vee \otimes M) \otimes (M^\vee \otimes M) &\cong \lambda^*(M^\vee) \otimes \lambda^*(M) \otimes M^\vee \otimes M \\
&\cong \lambda^*(M) \otimes M^\vee \otimes \lambda^*(M^\vee) \otimes M \\
&\cong (\lambda^*(M^\vee) \otimes M)^\vee \otimes \lambda^*(M^\vee) \otimes M \\
&\cong \mathrm{End}(\lambda^*(M^\vee) \otimes M).
\end{aligned}$$

Then the corresponding endomorphism of $\lambda^*(M^\vee) \otimes M$ is a projector of rank 1.

Proof. The construction of the Taylor isomorphism on modules satisfying the Robba condition is functorial, so the diagram

$$\begin{tikzcd} \lambda^*(\mathrm{End}(M)) \otimes \lambda^*(\mathrm{End}(M)) \arrow[r, "-\circ-"] \arrow[d, "T_\lambda \otimes T_\lambda"] & \lambda^*(\mathrm{End}(M)) \arrow[d, "T_\lambda"] \\ \mathrm{End}(M) \otimes \mathrm{End}(M) \arrow[r, "-\circ-"] & \mathrm{End}(M) \end{tikzcd}$$

commutes. From this, it follows formally that the endomorphism of $\lambda^*(M^\vee) \otimes M$ is a projector. The trace of this projector is an analytic function of λ, but is also equal to the rank of the projector and so always belongs to $\{0, \dots, \mathrm{rank}(M)\}$. It is thus a constant function; moreover, the constant value must equal 1 because for $\lambda = 1$, the endomorphism of $\lambda^*(M^\vee) \otimes M \cong \mathrm{End}(M)$ in question is the projector onto the trace component. This proves the claim. $\square$

3.3. **The Robba condition: residue characteristic** 0**.** We continue to study the Robba condition in the case of residue characteristic 0. The methods used are familiar, but the exact result seems to be inexplicably missing from the literature.

Hypothesis 3.3.1. Throughout §3.3, retain Hypothesis 3.2.1, but also assume that $p = 0$ and that M satisfies the Robba condition.

Definition 3.3.2. An *exponent* for M is a finite multisubset of $\mathfrak{o}_{K^{\mathrm{alg}}}$ such that $M[t^{-1}] \otimes_K K^{\mathrm{alg}}$ admits a basis on which D acts via a matrix over $\mathfrak{o}_{K^{\mathrm{alg}}}$ with multiset of eigenvalues equal to A.

Lemma 3.3.3. *Assume that* $0 \in I$ *and that the eigenvalues of* D *on* M/tM *belong to* $\mathfrak{o}_{K^{\mathrm{alg}}}$*. Then there exists a differential module* M' *over* R_I *with* $M[t^{-1}] \cong M'[t^{-1}]$ *such that the eigenvalues of* D *on* M'/tM' *belong to* $\mathfrak{o}_{K^{\mathrm{alg}}}$ *and are prepared.*

Proof. This is an example of the use of *shearing transformations* [25, Proposition 7.3.10]. Split M/tM as a direct sum in which each summand consists of the generalized eigenspaces for a single Galois orbit of eigenvalues for the action of D. If we consider the differential submodule M' of M consisting of those elements whose images in M/tM project to zero in a particular summand, the eigenvalues of D on M'/tM' are the same as on M/tM except that one Galois orbit has been shifted by 1.

It thus suffices to establish the existence of a sequence of shifts having the desired property. This follows from the following two observations (both of which require $p = 0$).

(a) If $\lambda, \lambda' \in \kappa_K^{\mathrm{alg}}$ are Galois conjugate and differ by an integer, then they are equal. (This follows by taking traces from some finite extension of κ_K containing λ, λ'.)
(b) If $\lambda_1, \lambda_1' \in \kappa_K^{\mathrm{alg}}$ are Galois conjugate, $\lambda_2, \lambda_2' \in \kappa_K^{\mathrm{alg}}$ are Galois conjugate, and $\lambda_1 - \lambda_2, \lambda_1' - \lambda_2' \in \mathbb{Z}$, then $\lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$. (This again follows by taking traces.)

$\square$

Lemma 3.3.4. *Assume that $0 \in I$ and the eigenvalues of D on M/tM belong to $\mathfrak{o}_{K^{\mathrm{alg}}}$ and are prepared. Then there exists a basis of M on which D acts via a matrix over $\mathfrak{o}_K$.*

Proof. Let $P(T) \in \mathfrak{o}_K[T]$ be the characteristic polynomial of the action of D on M/tM. Since the roots of P are prepared, for each positive integer j there exists a unique polynomial $Q_j(T) \in \mathfrak{o}_K[T]$ of degree at most $n-1$ such that $P(T-j)Q_j(T) \equiv 1 \pmod{P(T)}$.

It is straightforward to check (see for example [25, Proposition 7.3.6]) that there exists a basis of $M \otimes_{R_I} K[\![t]\!]$ on which D acts via a matrix over $\mathfrak{o}_K$. We may reconstruct this basis by starting with any elements $\mathbf{e}_1, \dots, \mathbf{e}_n \in M$ which lift a basis of M/tM and forming the t-adic limits of the sequences

$$\mathbf{e}_{i,m} = \left(\prod_{j=1}^{m} P(D-j)Q_j(D) \right) \mathbf{e}_i \qquad (i = 1, \dots, n; m = 1, 2, \dots).$$

For any given $\rho \in I - \{0\}$, these sequences are bounded for the norm induced by $|\cdot|_\rho$ using a basis of $M_{[0,\rho]}$ (because M satisfies the Robba condition); since these sequences also converge t-adically, they converge under $|\cdot|_{\rho'}$ for any $\rho' \in (0, \rho)$ by Lemma 3.1.5(b) (with $m = 1$). This proves the existence of the desired basis. $\square$

Lemma 3.3.5. *Assume that for some $\rho \in I$, M admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ on which D acts via a matrix $N = \sum_{i \in \mathbb{Z}} N_i t^i$ with $|N_0| \leq 1$ and $|N - N_0|_\rho < 1$ for all $\rho \in I$. Then there exists a basis of M on which D acts via a matrix over $\mathfrak{o}_K$.*

Proof. By applying Lemma 3.3.3 with K replaced by κ_K (equipped with the trivial norm) and using the fact that $\kappa_K[t^\pm]$ is a principal ideal domain (so every invertible square matrix over it factors as a product of elementary matrices), we may ensure that the eigenvalues of N_0 are prepared. In this case, for any nonzero $i \in \mathbb{Z}$, the eigenvalues of the linear operator $X \mapsto \overline{N_0}X - X\overline{N_0} + iX$ on $n \times n$ matrices over κ_K are all nonzero (because each of them has the form $\lambda - \lambda' + i$ for some eigenvalues λ, λ' of $\overline{N_0}$). Consequently, this linear operator is invertible;

it follows that for any $n \times n$ matrix X over K and any nonzero $i \in \mathbb{Z}$, $|N_0 X - X N_0 + iX| = |X|$.

We next produce a sequence $U_0, U_1, \dots$ of invertible matrices over R_I such that $|U_l - I_n|_\rho < 1$ for all $l \in \{0, 1, \dots\}$ and $\rho \in I$. Start with $U_0 = I_n$. Given U_l for some l, put $N_l = U_l^{-1} N U_l + U_l^{-1} D(U_l)$. Write $N_l = \sum_{i \in \mathbb{Z}} N_{l,i} t^i$ and apply the previous paragraph to construct X_l so that $|X_l|_\rho = |N_l - N_{l,0}|_\rho$ for all $\rho \in I$ and $N_{l,0} - N_l = X_l N_0 - N_0 X_l + D(X_l)$. Then put $V_l = I_n + X_l$ and $U_{l+1} = U_l V_l$; note that $N_{l+1} = V_l^{-1} N_l V_l + V_l^{-1} D(V_l)$.

Suppose that for $\rho \in I$ and $\epsilon > 0$, we have $|N - N_0|_\rho \leq \epsilon$ and $|N_l - N_{l,0}|_\rho \leq \epsilon^{l+1}$. We then have $|V_l - I_n|_\rho \leq \epsilon^{l+1}$, so

$$|N_{l+1} - N_l + X_l N_0 - N_0 X_l - D(X_l)|_\rho \leq \epsilon^{l+2}.$$

However, the matrix on the left side is exactly $N_{l+1} - N_{l,0}$, so we must have $|N_{l+1} - N_{l+1,0}|_\rho \leq \epsilon^{l+2}$.

From the previous paragraph, it follows that the U_l converge to an invertible matrix U over R_I. The elements $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ of M given by $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$ then form a basis with the desired property. □

Theorem 3.3.6. *Assume that $p = 0$ and that M satisfies the Robba condition.*

(a) *There exists a Galois-invariant exponent for M.*
(b) *Any two exponents of M are equivalent.*

Proof. Apply Corollary 2.1.6 to choose $\mathbf{v} \in M$ which is a cyclic vector for $M \otimes_{R_I} \mathrm{Frac}(R_I)$. For any closed subinterval J of I of positive length, the quotient of M_J by the span of $\mathbf{v}, D(\mathbf{v}), \dots, D^{n-1}(\mathbf{v})$ is killed by some nonzero element of R_J; since the slopes of the Newton polygon of this element form a discrete subset of J, we can shrink J so as to force this element to become a unit. That is, we may choose J so that $\mathbf{v}, D(\mathbf{v}), \dots, D^{n-1}(\mathbf{v})$ form a basis of M_J.

Let N be the matrix of action of D on the basis $\mathbf{v}, D(\mathbf{v}), \dots, D^{n-1}(\mathbf{v})$ of M_J. By Proposition 2.2.6, we have $|N|_J \leq 1$. In particular, if we write $N = \sum_{i \in \mathbb{Z}} N_i t^i$, then $|N_0| \leq 1$. Since J has positive length and $|N - N_0|_J \leq 1$, by shrinking J and applying Lemma 3.1.5(b) (with $m = 1$) we may ensure that $|N - N_0|_J < 1$. We may then apply Lemma 3.3.5 to obtain the conclusion of (a) for M_J. We may then use Lemma 3.3.3 and Lemma 3.3.4 to extend the convergence from J to I. This yields (a). Given (a), (b) follows from the fact that M_λ is trivial if and only if $\lambda \in \mathbb{Z}$. □

3.4. **The Robba condition: residue characteristic $p > 0$.** When $p > 0$, the structure of modules satisfying the Robba condition is more

complicated; it is best understood using the Christol-Mebkhout theory of p-adic exponents. Here we follow and refine the exposition in [25, Chapter 13].

Hypothesis 3.4.1. Throughout §3.4, retain Hypothesis 3.2.1, but also assume that $p > 0$ and that M satisfies the Robba condition.

Definition 3.4.2. We say $a \in \mathbb{Z}_p$ is a *p-adic Liouville number* if $a \notin \mathbb{Z}$ and

$$\liminf_{m \to \infty} \frac{p^m}{m} \left\langle \frac{a}{p^m} \right\rangle < +\infty. \tag{3.4.2.1}$$

Otherwise, we say a is a *p-adic non-Liouville number*.

For A a multisubset of $\mathbb{Z}_p$, we say that A is *p-adic non-Liouville* if it contains no p-adic non-Liouville number. We say that A has *p-adic non-Liouville differences* if the *difference multiset* of A, defined as

$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\},$$

is p-adic non-Liouville.

Definition 3.4.3. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be two finite multisubsets of $\mathbb{Z}_p$ of the same cardinality n. We say that A and B are *weakly equivalent* if there exist a constant $c > 0$ and a sequence $\sigma_1, \sigma_2, \dots$ of permutations of $\{1, \dots, n\}$ such that

$$p^m \left\langle \frac{a_{\sigma_m(i)} - b_i}{p^m} \right\rangle \leq cm \qquad (m = 1, 2, \dots; \, i = 1, \dots, n).$$

This is evidently an equivalence relation. Note that A, B are weakly equivalent if they are equivalent in the sense of Definition 3.2.4; the converse is false in general (see [25, Example 13.4.6]) but is true for $n = 1$ (see Corollary 3.4.7 below).

All of the key properties of weak equivalence can be expressed in terms of the following construction.

Definition 3.4.4. Let $A, A_1, \dots, A_k$ be multisubsets of $\mathbb{Z}_p$ such that A is the multiset union of $A_1, \dots, A_k$. We say that $A_1, \dots, A_k$ form an *integer partition* (resp. a *Liouville partition*) of A if there do not exist distinct values $g, h \in \{1, \dots, k\}$ and elements $a_g \in A_g, a_h \in A_h$ such that $a_g - a_h$ is an integer (resp. an integer or a p-adic Liouville number). This implies in particular that A_g and A_h are disjoint, so $A_1, \dots, A_k$ form a partition of A.

Note that A always admits a maximal integer partition, namely the partition into $\mathbb{Z}$-cosets. This partition is a Liouville partition if and only if A has p-adic non-Liouville differences.

Proposition 3.4.5. *Let A be a finite multisubset of $\mathbb{Z}_p$ and let $A_1, \dots, A_k$ be a Liouville partition of A.*

(a) *Let $B_1, \dots, B_k$ be multisubsets of $\mathbb{Z}_p$ such that B_g is weakly equivalent to A_g for $g = 1, \dots, k$. Then $B_1, \dots, B_k$ form a Liouville partition of $B = B_1 \cup \cdots \cup B_k$; in particular, $B_1, \dots, B_k$ are pairwise disjoint.*

(b) *Suppose B is a multisubset of $\mathbb{Z}_p$ weakly equivalent to A. Then B admits a Liouville partition $B_1, \dots, B_k$ such that B_g is weakly equivalent to A_g for $g = 1, \dots, k$.*

Proof. By the conditions on A, for each $c > 0$, there exists $m_0 = m_0(c)$ such that for all $m \geq m_0$, $g, h \in \{1, \dots, k\}$ with $g \neq h$, $a_g \in A_g$, $a_h \in A_h$,

$$p^m \left\langle \frac{a_g - a_h}{p^m} \right\rangle > (3c+1)m. \tag{3.4.5.1}$$

Assume now the hypotheses of (a). Suppose by way of contradiction that there exist $g, h \in \{1, \dots, k\}$ with $g \neq h$, $b_g \in B_g$, $b_h \in B_h$ such that $b_g - b_h$ is an integer or a p-adic Liouville number. Then there exists $c > 0$ such that for each m, on one hand

$$p^m \left\langle \frac{b_g - b_h}{p^m} \right\rangle \leq cm$$

and on the other hand there exist $a_g \in A_g$, $a_h \in A_h$ such that

$$p^m \left\langle \frac{a_g - b_g}{p^m} \right\rangle, p^m \left\langle \frac{a_h - b_h}{p^m} \right\rangle \leq cm.$$

But then

$$p^m \left\langle \frac{a_g - a_h}{p^m} \right\rangle \leq 3cm,$$

which combined with (3.4.5.1) yields the desired contradiction.

Assume now the hypotheses of (b). Label the elements of A and B as $a_1, \dots, a_n$ and $b_1, \dots, b_n$, respectively. Then there exists $c > 0$ such that for each m, there exists a permutation σ_m of $\{1, \dots, n\}$ such that

$$p^m \left\langle \frac{a_{\sigma_m(i)} - b_i}{p^m} \right\rangle \leq cm \qquad (i = 1, \dots, n).$$

In particular,

$$p^m \left\langle \frac{a_{\sigma_m(i)} - a_{\sigma_{m+1}(i)}}{p^m} \right\rangle \leq (2c+1)m \qquad (i = 1, \dots, n),$$

which by (3.4.5.1) yields that for $m \geq m_0(c)$, $\sigma_m^{-1} \circ \sigma_{m+1}$ must respect the partition of A. Define $B_1, \dots, B_k$ so that B_g consists of those b_i

for which $a_{\sigma_{m)}(i)} \in A_g$ for $m \geq m_0(c)$; by the above argument, B_g is weakly equivalent to A_g. By (a), $B_1, \dots, B_k$ is a Liouville partition of B, as desired. □

Corollary 3.4.6. *Let A, B be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Then A contains an integer or a p-adic non-Liouville number if and only if B does.*

Proof. Note that A contains an integer or p-adic non-Liouville number if and only if $\{0\}$ and A fail to form a Liouville partition of $\{0\} \cup A$. The claim thus follows by applying Proposition 3.4.5(a) to $\{0\} \cup A$ and $\{0\} \cup B$. □

The following corollary reproduces [25, Lemma 13.4.3].

Corollary 3.4.7. *For $a, b \in \mathbb{Z}_p$, the singleton multisets $\{a\}, \{b\}$ are weakly equivalent if and only if $a - b \in \mathbb{Z}$.*

Proof. By translating both a and b, we may assume $b = 0$. If $a \in \mathbb{Z}$, then $\{a\}$ and $\{0\}$ are equivalent and hence weakly equivalent. Conversely, if $\{a\}$ and $\{0\}$ are weakly equivalent, then a satisfies (3.4.2.1) and so must be either an integer or a p-adic Liouville number, but the latter case is ruled out by Corollary 3.4.6. □

The following corollary reproduces [25, Proposition 13.4.5].

Corollary 3.4.8. *Let A, B be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Suppose that A has p-adic non-Liouville differences. Then A and B are equivalent.*

Proof. By partition A into $\mathbb{Z}$-cosets and applying Proposition 3.4.5(b), we may reduce to the case where A is a multisubset of $\mathbb{Z}$. In this case, for each $b \in B$, the singleton multisets $\{0\}$ and $\{b\}$ are weakly equivalent, so Corollary 3.4.7 implies that $b \in \mathbb{Z}$. This proves the claim. □

Corollary 3.4.9. *Let A, B be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Suppose that A is p-adic non-Liouville. Then there exist Liouville partitions A_1, A_2 of A and B_1, B_2 of B satisfying the following conditions.*

(a) *The multisets A_1, B_1 consist entirely of integers.*
(b) *The multisets A_2, B_2 are weakly equivalent and contain no integers or p-adic Liouville numbers.*

In particular, B is also p-adic non-Liouville.

Proof. Partition A into two parts A_1, A_2 so that A_1 consists precisely of the integers appearing in A; by hypothesis, this is a Liouville partition of A. By Proposition 3.4.5(b), B admits a Liouville partition B_1, B_2 in which B_i is weakly equivalent to A_i for $i = 1, 2$. Since A_1 consists only of integers, by Corollary 3.4.8, B_1 also consists only of integers. Since A_2 does not contain any integer or p-adic Liouville number, neither does B_2 by Corollary 3.4.6. This proves the desired results. □

Corollary 3.4.10. *Let A be a finite multisubset of $\mathbb{Z}_p$ such that $A - A$ is weakly equivalent to a p-adic non-Liouville multiset. Then A has p-adic non-Liouville differences.*

Proof. By Corollary 3.4.9, $A - A$ is p-adic non-Liouville. □

Definition 3.4.11. Recall that we are assuming that M satisfies the Robba condition. Let J be a closed subinterval of I of positive length. We say that the multisubset $A = \{a_1, \dots, a_n\}$ of $\mathbb{Z}_p$ is an *exponent* for M over J if there exist elements $\mathbf{v}_{m,A,j} \in M_J[t^{-1}]$ for $m = 1, 2, \dots$ and $j = 1, \dots, n$ satisfying the following conditions. (For this definition, we fix an ordering of A, but this choice is manifestly immaterial.)

(a) For all m, j, for all $\zeta \in K^{\mathrm{alg}}$ with $\zeta^{p^m} = 1$, we have $\zeta^*(\mathbf{v}_{m,A,j}) = \zeta^{a_j}\mathbf{v}_{m,A,j}$ as an equality in $M_J[t^{-1}] \otimes_K K(\zeta)$.

(b) For some (and hence any) basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of M_J, there exists $k > 0$ such that the $n \times n$ matrices $S_{m,A}$ over R_J defined by $\mathbf{v}_{m,A,j} = \sum_i (S_{m,A})_{ij}\mathbf{e}_i$ are invertible and satisfy

$$\left|S_{m,A}\right|_J, \left|S_{m,A}^{-1}\right|_J \leq p^{km} \qquad (m = 1, 2, \dots).$$

Note that if A is an exponent for M over J, then so is any multiset equivalent to A (but not necessarily any multiset weakly equivalent to A).

Remark 3.4.12. In [25, Definition 13.5.2], the hypotheses on the matrix $S_{m,A}$ are slightly different: it is assumed that $S_{m,A}$ is invertible and satisfies $\left|S_{m,A}\right|_J \leq p^{km}$ and $\left|\det(S_{m,A})\right|_J \geq 1$. It is easy to see that this hypothesis is equivalent to the one given in Definition 3.4.11 modulo rescaling the vectors $\mathbf{v}_{m,A,j}$ and rechoosing the constant k; we may thus safely quote results from [25] in what follows.

Example 3.4.13. As noted in Example 3.2.3, for any $\lambda \in \mathbb{Z}_p$, the differential module M_λ generated by a single element $\mathbf{v}$ satisfying $D(\mathbf{v}) = \lambda\mathbf{v}$ satisfies the Robba condition [25, Example 9.5.2]. This module admits the singleton multiset $\{\lambda\}$ as an exponent.

Remark 3.4.14. If M_1, M_2 are two differential modules over R_I for the derivation $t\frac{d}{dt}$ admitting respective exponents A_1, A_2 over some J, we then have the following.

(a) If there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of differential modules over R_I, then M admits the multiset union $A_1 \cup A_2$ as an exponent over J.
(b) The differential module $M_1 \otimes M_2$ admits the multiset$A_1 + A_2 = \{a_i + a_j : a_i \in A_1, a_j \in A_2\}$ as an exponent over J.
(c) The differential module $M_1^\vee$ admits the multiset $-A_1 = \{-a : a \in A_1\}$ as an exponent over J.

Remark 3.4.15. If $0 \in I$, then it is straightforward to check the following by imitating the proof of [25, Theorem 13.2.2].

(a) Let A be the set of eigenvalues of D on M/tM. Then A belongs to $\mathbb{Z}_p^n$ and is an exponent for M.
(b) Any Liouville partition of A corresponds to a unique direct sum decomposition of M.
(c) If A is a multisubset of $\lambda + \mathbb{Z}$, then there exists another differential module M' over R_I with $M[t^{-1}] \cong M'[t^{-1}]$ such that D acts on M'/tM' via a matrix with all eigenvalues equal to λ. (This again follows from the use of shearing transformations as in Lemma 3.3.3.)
(d) If A is a multisubset of $\{\lambda\}$, then there exists a basis of M on which D acts via a matrix over K with all eigenvalues equal to λ.

We may thus safely assume $0 \notin I$ in what follows.

Theorem 3.4.16. (a) *For any closed subinterval J of I of positive length not containing* 0*, there exists an exponent for M over J.*
(b) *Any two exponents for M (possibly over different intervals) are weakly equivalent.*

Proof. For (a), see [25, Theorem 13.5.5]. For (b), let J_1, J_2 be two closed subintervals of I of positive length not containing 0. Let A_1, A_2 be exponents for M over J_1, J_2. If $J_1 = J_2$, we may apply [25, Theorem 13.5.6] to deduce that A_1 is weakly equivalent to A_2. Otherwise, let J be a third such interval containing both J_1 and J_2. By (a), there exists an exponent A for M over J, which then restricts to an exponent for M over J_1 and over J_2. By [25, Theorem 13.5.6] again, A is weakly equivalent to both A_1 and A_2, so A_1 and A_2 are weakly equivalent to each other. □

Remark 3.4.17. In case M admits a basis (e.g., if K is spherically complete), the proofs of [25, Theorem 13.5.5, Theorem 13.5.6] show that the exponent A and the elements $\mathbf{v}_{m,A,j}$ can be chosen uniformly in J. We will not need to use this fact in this paper.

Definition 3.4.18. We say that M has *p-adic non-Liouville exponents* if for some closed subinterval J of I of positive length not containing 0, M admits an exponent A over J which is p-adic non-Liouville. By Theorem 3.4.16(b) and Corollary 3.4.9, this implies that every exponent of M (over every J) is p-adic non-Liouville.

We say that M has *p-adic non-Liouville exponent differences* if $\mathrm{End}(M)$ has p-adic non-Liouville exponents. For alternate characterizations, see Lemma 3.4.19 below.

Lemma 3.4.19. *The following conditions are equivalent.*

(a) *The module M has p-adic non-Liouville exponent differences.*
(b) *Some exponent of M has p-adic non-Liouville differences.*
(c) *Every exponent of M has p-adic non-Liouville differences.*

Moreover, when these conditions hold, then any two exponents of M are equivalent (not just weakly equivalent).

Proof. By Theorem 3.4.16(a), (c) implies (b). By Remark 3.4.14, (b) implies (a).

Suppose now that (a) holds. Let A be any exponent for M, and let B be an exponent for $\mathrm{End}(M)$ which is p-adic non-Liouville. By Remark 3.4.14, $A-A$ is an exponent for $\mathrm{End}(M)$, so by Theorem 3.4.16(b), $A-A$ and B are weakly equivalent. By Corollary 3.4.10, A has p-adic non-Liouville differences, yielding (c). Moreover, if A' is another exponent for M, then A and A' are weakly equivalent by Theorem 3.4.16(b), so A and A' are equivalent by Corollary 3.4.8. $\square$

The primary structure theorem for differential modules satisfying the Robba condition is the decomposition theorem of Christol-Mebkhout; see for instance [25, Theorem 13.6.1] and the errata to [25]. Here, we divide the statement into two parts in order to clarify the exposition and strengthen one of the two parts. One of the two parts, which by itself is sufficient for many applications, is the following structure theorem for modules admitting a singleton exponent.

Theorem 3.4.20. *Suppose that M admits an exponent identically equal to some $\lambda \in \mathbb{Z}_p$. Then for any closed subinterval J of I of positive length, M_J admits a basis on which D acts via a matrix over K whose eigenvalues are all equal to λ.*

Proof. We may assume $0 \notin I$ thanks to Remark 3.4.15. By replacing M with its twist $M_\lambda^\vee \otimes M$, we may reduce the theorem to the special case $\lambda = 0$. Let J be any closed subinterval of I of positive length; by Lemma 3.4.19, the zero n-tuple is an exponent for M over J. Choose $\eta > 1$ and $\alpha, \alpha', \beta, \beta' \in I$ such that $\alpha' < \beta'$, $\alpha' = \alpha\eta$, $\beta' = \beta/\eta$, and

$J \subseteq [\alpha', \beta']$. Fix a basis of M_J and define the matrices $S_{m,A}$ as in Definition 3.4.11 for $A = \{0, \dots, 0\}$. Choose $\lambda \in (0,1)$ and $c > 0$ so that $p^{10k}\eta^{-c} \leq \lambda$, then choose $m_0 > 0$ so that $p^m > cm$ for all $m \geq m_0$. We will construct invertible matrices R_m over K for $m \geq m_0$ such that $R_{m_0} = I_n$ and

$$\left| I_n - R_m S_{m,A}^{-1} S_{m+1,A} R_{m+1}^{-1} \right|_\rho \leq \lambda^m \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

This will imply that for $m \geq m_0$ and $\rho \in [\alpha', \beta']$, $\left| I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1} \right|_\rho < 1$ and $\left| S_{m_0,A}^{-1} S_{m,A} R_m^{-1} - S_{m_0,A}^{-1} S_{m+1,A} R_{m+1}^{-1} \right|_\rho \leq \lambda^m$. Consequently, the sequence $S_{m_0,A}^{-1} S_{m,A} R_m^{-1}$ for $m = m_0, m_0+1, \dots$ will converge to an invertible matrix U over $R_{[\alpha',\beta']}$ such that $S_{m_0,A}U$ is the change-of-basis matrix to a basis of $M_{[\alpha',\beta']}$ of the desired form. This will complete the proof.

The construction of the R_m proceeds recursively as follows. Given $R_{m_0}, \dots, R_m$, we first verify that

$$|R_m|, \left|R_m^{-1}\right| \leq p^{2km}.$$

This is clear for $m = m_0$, so we may assume $m > m_0$. Choose any $\rho \in [\alpha', \beta']$. As noted above, we have $\left| I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1} \right|_\rho < 1$, so $\left| S_{m_0,A}^{-1} S_{m,A} R_m^{-1} \right|_\rho = \left| R_m S_{m,A}^{-1} S_{m_0,A} \right|_\rho = 1$. We then deduce the claim from the bound $|S_{m,A}|_\rho, \left|S_{m,A}^{-1}\right|_\rho \leq p^{km}$.

Next, put $T_m = R_m S_{m,A}^{-1} S_{m+1,A}$; we then have

$$|T_m|_{[\alpha,\beta]}, \left|T_m^{-1}\right|_{[\alpha,\beta]} \leq p^{4km+k}.$$

Let $T_{m,0}$ be the constant coefficient of T_m. Since T_m is a series in t^{p^m}, Lemma 3.1.5(b) implies

$$|T_m - T_{m,0}|_{[\alpha',\beta']} \leq p^{4km+k}\eta^{-p^m}.$$

We may now take $R_{m+1} = T_{m,0}$, because

$$\begin{aligned} \left| I_n - R_{m+1} T_m^{-1} \right|_{[\alpha',\beta']} &\leq \left| T_m^{-1} \right|_{[\alpha',\beta']} \cdot |T_m - T_{m,0}|_{[\alpha',\beta']} \\ &\leq p^{8km+2k}\eta^{-p^m} \\ &< p^{10km}\eta^{-cm} \leq \lambda^m < 1 \end{aligned}$$

and so $\left| I_n - T_m R_{m+1}^{-1} \right|_{[\alpha',\beta']} \leq \lambda^m$. This completes the construction of the R_m and thus the proof. $\square$

Remark 3.4.21. Theorem 3.4.20 is sufficient to recover the full Christol-Mebkhout decomposition theorem in the case of a differential module admitting an exponent contained in $\mathbb{Z}_p \cap \mathbb{Q}$, by pulling back along the map $t \mapsto t^m$ for a suitably divisible integer $m \in \mathbb{Z}$.

The second part is a splitting theorem for modules admitting an exponent with p-adic non-Liouville differences. This may be generalized as follows.

Theorem 3.4.22. *Suppose that M admits an exponent A admitting the Liouville partition $A_1, \dots, A_k$. Then for any closed subinterval J of I of positive length, there exists a unique direct sum decomposition $M_J = M_1 \oplus \cdots \oplus M_k$ such that for $g = 1, \dots, k$, M_g admits an exponent over J weakly equivalent to A_g.*

Proof. We may assume $0 \notin I$ thanks to Remark 3.4.15. We first verify uniqueness. Suppose to the contrary that there is a second decomposition $M_J = M'_1 \oplus \cdots \oplus M'_k$ of the desired form for which there exist $g \neq h$ such that $M_{gh} = M_g \cap M'_h$ is nonzero. Apply Theorem 3.4.16(a) to produce exponents B_1, B_2, B_3 of M_{gh}, M_g/M_{gh}, M'_h/M_{gh}. By Remark 3.4.14 and Theorem 3.4.16(b), $B_1 \cup B_2$ is weakly equivalent to A_g and $B_1 \cup B_3$ is weakly equivalent to A_h. We then obtain the desired contradiction by applying Proposition 3.4.5(a).

We next verify existence. To simplify notation, we may reduce to the case $k = 2$. Let J be any closed subinterval of I of positive length; by Theorem 3.4.16(a,b) and Proposition 3.4.5(b), M admits an exponent A over J of the specified form. Choose an ordering $A = \{a_1, \dots, a_n\}$. Choose $\eta > 1$ and $\alpha, \alpha', \beta, \beta' \in I$ such that $\alpha' < \beta'$, $\alpha' = \alpha\eta$, $\beta' = \beta/\eta$, and $J \subseteq [\alpha', \beta']$. Fix a basis of M_J and define the matrices $S_{m,A}$ as in Definition 3.4.11. Choose $\lambda \in (0, 1)$ and $c > 0$ so that $p^{9k}\eta^{-c} \leq \lambda$. By hypothesis, there exists $m_0 > 0$ such that for $m \geq m_0$, $b_1 \in A_1$, $b_2 \in A_2$, the congruence $h \equiv b_1 - b_2 \pmod{p^m}$ forces $|h| \geq cm$.

Let Π_m be the projector onto the submodule of M_J generated by $\mathbf{v}_{m,A,i}$ for those i for which $a_i \in A_1$; then

$$|\Pi_m|_{[\alpha,\beta]} \leq p^{2km}.$$

For those j for which $a_j \in A_1$, write $(\Pi_m - \Pi_{m+1})(\mathbf{v}_{m,A,j}) = \sum_i a_{m,i}\mathbf{v}_{m+1,A,i}$, so that

$$|a_{m,i}|_{[\alpha,\beta]} \leq p^{4km+2k}.$$

Since $(\Pi_m - \Pi_{m+1})(\mathbf{v}_{m,A,j}) = (1 - \Pi_{m+1})(\mathbf{v}_{m,A,j})$, we have $a_{m,i} = 0$ when $a_i \in A_1$. On the other hand, when $a_i \in A_2$, the coefficient of t^h in $a_{m,i}$ can only be nonzero if $h \equiv a_i - a_j \pmod{p^m}$; this implies

$$|a_{m,i}|_{[\alpha',\beta']} \leq p^{4km+2k}\eta^{-cm} \qquad (m \geq m_0)$$

by Lemma 3.1.5(a), and so

$$|(\Pi_m - \Pi_{m+1})(\mathbf{v}_{m,A,j})|_{[\alpha',\beta']} \leq p^{5km+2k}\eta^{-cm} \qquad (m \geq m_0).$$

Similarly, for those j for which $a_j \in A_2$,

$$|(\Pi_m - \Pi_{m+1})(\mathbf{v}_{m,A,j})|_{[\alpha',\beta']} \leq p^{5km+3k}\eta^{-cm} \qquad (m \geq m_0)$$

and so

$$|(\Pi_m - \Pi_{m+1})|_{[\alpha',\beta']} \leq p^{6km+3k}\eta^{-cm} \leq \lambda^m \qquad (m \geq m_0).$$

Therefore the Π_m converge to an endomorphism of M_J, which is forced to be a projector defining the desired splitting. □

We may put Theorem 3.4.20 and Theorem 3.4.22 to separate integer exponents from p-adic non-Liouville exponents.

Corollary 3.4.23. *Suppose that M has p-adic non-Liouville exponents. Then there exists a unique direct sum decomposition $M \cong M_1 \oplus M_2$ with the following properties.*

(a) *The module $M_1[t^{-1}]$ admits a basis on which D acts via a nilpotent matrix over K. In particular, $M_1[t^{-1}]$ is unipotent (i.e., it is a successive extension of trivial differential modules over $R_I[t^{-1}]$).*

(b) *No exponent of M_2 contains an integer or a p-adic Liouville number.*

Proof. We obtain the splitting $M \cong M_1 \oplus M_2$ using Theorem 3.4.22. We then obtain (a) using Theorem 3.4.22 and (b) using Corollary 3.4.6. □

We recover as a corollary the original decomposition theorem of Christol-Mebkhout, as stated in [25, Theorem 13.6.1].

Corollary 3.4.24. *Fix a set S of coset representatives of $\mathbb{Z}$ in $\mathbb{Z}_p$. Suppose that M has p-adic non-Liouville exponent differences. Then there exists a unique direct sum decomposition $M \cong \bigoplus_{\lambda \in S} N_\lambda$ in which N_λ admits a basis on which D acts via a matrix over K with all eigenvalues equal to λ. In particular, N_λ is isomorphic to a successive extension of copies of M_λ.*

Proof. We induct on $\operatorname{rank}(M)$. By twisting, we can force M to admit an exponent containing 0. We may then split M using Corollary 3.4.23 and continue. □

Corollary 3.4.25. *If $M^{\otimes p}$ is unipotent (resp. trivial), then so is M.*

Proof. We first prove the unipotent case. Apply Theorem 3.4.16(a) to construct an exponent A for M. By Remark 3.4.14, the p-fold sum $A + \cdots + A$ is an exponent for $M^{\otimes p}$, as by assumption is the zero tuple. Since the latter has p-adic non-Liouville differences, Lemma 3.4.19 implies that $A + \cdots + A$ is equivalent (not just weakly equivalent) to 0. In

particular, $pa \in \mathbb{Z}$ for each $a \in A$; since $a \in \mathbb{Z}_p$, this is only possible when $a \in \mathbb{Z}$ for each $a \in A$. By Corollary 3.4.23, M is unipotent.

Suppose now that $M^{\otimes p}$ is trivial. By the previous paragraph, M admits a basis on which D acts via a nilpotent matrix N over K; note that the conjugacy class of the matrix N is uniquely determined by M because 1 is nonzero in the cokernel of $t\frac{d}{dt}$ on R_I. In particular, since $M^{\otimes p}$ is trivial, the p-th tensor power of N must be zero, but this is only possible if N is itself zero. Consequently, M itself must be trivial. $\square$

Remark 3.4.26. It is known that Theorem 3.4.22 does not extend to integer partitions. For example, if λ is a p-adic Liouville number, then there exist nontrivial extensions of M_0 by M_λ, each of which admits $\{0, \lambda\}$ as an exponent by Remark 3.4.14. In fact, we expect (although we have no examples) that one can even find irreducible differential modules of rank greater than 1 satisfying the Robba condition.

3.5. **Frobenius antecedents and descendants.** The construction of Frobenius antecedents and descendants can be generalized to differential modules over power series. We record here some key facts from [25, Chapter 10] which we will use.

Hypothesis 3.5.1. Throughout §3.6, assume $p > 0$, fix a subinterval I of $[0, +\infty)$, and take $R = R_I$ or $R = R_I^{\mathrm{an}}$. Let (M, D) be a differential module of rank n over $(R, \frac{d}{dt})$.

Definition 3.5.2. Let I^p be the subinterval of $[0, +\infty)$ consisting of γ^p for all $\gamma \in I$. In case $R = R_I$ (resp. $R = R_I^{\mathrm{an}}$), let R' be a copy of R_{I^p} (resp. $R_{I^p}^{\mathrm{an}}$) in the variable t^p, identified with a subring of R. We may then view $(R', \frac{d}{dt^p})$ as a differential ring.

If $0 \notin I$, we may form the *Frobenius descendant* $\varphi_* M$ as in [25, Definition 10.3.4]; that is, $\varphi_* M$ is a copy of M equipped with the derivation $D' = p^{-1}t^{1-p}D$. For any $\rho \in I$, $(\varphi_* M) \otimes_S F'_\rho$ may be naturally identified with the Frobenius descendant of M_ρ.

Proposition 3.5.3. *Suppose that $f_i(M, r) < r - \log \omega$ for all $r \in -\log I$. Then there exists a unique (up to unique isomorphism) differential module M' over $(R', \frac{d}{dt^p})$ such that for $\rho \in I \setminus \{0\}$, $M' \otimes_{R'} F'_\rho$ is the Frobenius antecedent of M_ρ.*

Proof. See [25, Theorem 10.4.4]. $\square$

We will also need to consider "off-center Frobenius descendants" as in [25, §10.8].

Definition 3.5.4. Suppose that $R = R_{[0,1]}^{\mathrm{an}}$. Let R'' be a copy of $R_{[0,1]}^{\mathrm{an}}$ in the variable u. For $\rho \in (0, 1]$, let F''_ρ be a copy of F_ρ in the variable

u, so that R'' maps to F''_ρ. Choose $\lambda \in K$ with $|\lambda| = 1$, then identify R'' with a subring of R by identifying u with $(t+\lambda)^p - \lambda^p$.

Proposition 3.5.5. *Suppose that $R = R^{\mathrm{an}}_{[0,1]}$. Let $\psi_* M$ be a copy of M viewed as a differential module over $(R'', \frac{d}{du})$. Then for $\rho \in (\omega, 1]$, the multiset consisting of the intrinsic subsidiary radii of $(\psi_* M) \otimes_{R''} F''_{\rho^p}$ is the union of the multisets*

$$\begin{cases} \{s^p\} \cup \{\omega^p \rho^{-p} \text{ } (p-1 \text{ times})\} & \text{if } s > \omega\rho \\ \{p^{-1} s \rho^{1-p} \text{ } (p \text{ times})\} & \text{if } s \leq \omega/\rho \end{cases}$$

for s running over the intrinsic subsidiary radii of M_ρ.

Proof. By rescaling t, we may reduce to the case where $\lambda = 1$. In this case, see [25, Theorem 10.8.3]. □

3.6. **Variation of intrinsic radii.** We now consider differential modules not necessarily satisfying the Robba condition, with an eye towards the variation of the intrinsic subsidiary radii. The results we report here are taken from [25, Chapters 11–12]; starting in §4, we will see how to make more definitive statements in the language of Berkovich spaces. To facilitate this transition, we record a couple of important direct corollaries of the results of [25].

Hypothesis 3.6.1. Throughout §3.6, fix a subinterval I of $[0, +\infty)$, and let M be a differential module of rank n over $(R^{\mathrm{an}}_I, \frac{d}{dt})$.

Definition 3.6.2. For $\rho \in I \setminus \{0\}$, put $M_\rho = M \otimes_{R^{\mathrm{an}}_I} F_\rho$. For $r \in -\log I$ and $i = 1, \dots, n$, define $f_i(M, r)$ so that the list of intrinsic subsidiary radii of $M_{e^{-r}}$ in increasing order is

$$\exp(r - f_1(M, r)), \dots, \exp(r - f_n(M, r)).$$

Put $F_i(M, r) = f_1(M, r) + \cdots + f_n(M, r)$. As observed in Definition 2.2.2, the functions f_i and F_i are invariant under enlargement of the constant field K.

Proposition 3.6.3. *For $i = 1, \dots, n$, we have the following.*

(a) *(Linearity) The functions $f_i(M, r)$ and $F_i(M, r)$ are continuous and piecewise affine. Moreover, these functions assume only finitely many different slopes over any closed subinterval of $-\log I$ (even if $0 \in I$).*
(b) *(Integrality) If $i = n$ or $f_i(M, r_0) > f_{i+1}(M, r_0)$, then the slopes of $F_i(M, r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(M, r)$ and $F_i(M, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$.*

(c) *(Subharmonicity) Suppose that K is algebraically closed, $\alpha < 1 < \beta$, and $f_i(M,0) > 0$. Let $s_{\infty,i}(M)$ and $s_{0,i}(M)$ be the left and right slopes of $F_i(M,r)$ at $r = 0$. For $\overline{\mu} \in \kappa_K^\times$, choose any $\mu \in \mathfrak{o}_K$ lifting $\overline{\mu}$, let T_μ denote the substitution $t \mapsto t + \mu$, and let $s_{\overline{\mu},i}(M)$ be the right slope of $F_i(T_\mu^*(M), r)$ at $r = 0$. Then*

$$s_{\infty,i}(M) \leq \sum_{\overline{\mu} \in \kappa_K} s_{\overline{\mu},i}(M),$$

with equality if either $i = n$ and $f_n(M,0) > 0$, or $i < n$ and $f_i(M,0) > f_{i+1}(M,0)$.

(d) *(Monotonicity) Suppose that $0 \in I$. Then for any point r_0 where $f_i(M,r_0) > r_0$, the slopes of $F_i(M,r)$ are nonpositive in some neighborhood of r_0.*

(e) *(Convexity) The function $F_i(M,r)$ is convex.*

Proof. See [25, Theorem 11.3.2]. ☐

Corollary 3.6.4. *Suppose that $0 \in I$ and $f_1(M,r_0) = r_0$ for some $r_0 \in -\log I$. Then $f_1(M,r) = r$ for all $r \geq r_0$.*

Proof. By Proposition 3.6.3(a,d,e), the function $f_1(M,r)$ is piecewise affine and convex everywhere, and nonincreasing wherever it is greater than r. Since $f_1(M,r_0) = r_0$ and $f_1(M,r) \geq r$ everywhere, all of the slopes of $f_1(M,r)$ for $r \geq r_0$ must be at least 1. However, none of them can be strictly greater than 1 because this would force $f_1(M,r) > r$ for some r, and then $f_1(M,r)$ would be forced to be nonincreasing. This proves the claim. ☐

Corollary 3.6.5. *Suppose that $0 \in I$ and for some $r_0 \in -\log I$, some $r_1 > r_0$, and some $j \in \{0, \dots, n\}$, the functions $f_1(M,r), \dots, f_j(M,r)$ are equal to some constant value c for $r \in (r_0, r_1)$. Then*

$$f_i(M,r) = \max\{r, c\} \qquad (r > r_0;\ i = 1, \dots, j).$$

Proof. We prove that the claim holds for $f_1, \dots, f_i$ by induction on i, with base case $i = 0$. Given the induction hypothesis for $i - 1$, note that since $f_i(M,r) \geq r$ for all $r > r_0$, we must have $c > r_0$. By Proposition 3.6.3(d,e) and the induction hypothesis, the function $f_i(M,r)$ is convex everywhere and nonincreasing wherever it is greater than r. It follows that $f_i(M,r) = c$ for $r \in (r_0, c]$. By Corollary 3.6.4, we then have $f_i(M,r) = r$ for $r \geq c$. ☐

Proposition 3.6.6. *For $i = 1, \dots, n$, on any interval where $f_i(M,r)$ is affine, it has the form $ar + b$ for some $a \in \mathbb{Q}$ and some b in the divisible closure of $\log |K^\times|$.*

Proof. See [25, Corollary 11.8.2]. □

Proposition 3.6.7. *Suppose that $0 \in I$ and that for some $i \in \{1, \dots, n-1\}$ and some $\gamma \in I \setminus \{0\}$, the following conditions hold.*

(a) *The function $F_i(M, r)$ is constant for $r < -\log \gamma$.*
(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $r < -\log \gamma$.*

Then M admits a unique direct sum decomposition separating the first i intrinsic subsidiary radii of M_ρ for all $\rho > \gamma$.

Proof. See [25, Theorem 12.5.1]. □

Corollary 3.6.8. *Suppose that $I = [0, \beta]$ or $I = [0, \beta)$ for some $\beta > 0$ and put $r_0 = -\log \beta$. Suppose that for some $i \in \{0, \dots, n\}$ and some $r_1 > r_0$, the following conditions hold.*

(a) *For $j = 1, \dots, i$, the function $f_j(M, r)$ is constant for $r \in (r_0, r_1)$.*
(b) *If $i < n$, then $\liminf_{r \to r_0^+} f_{i+1}(M, r) = r_0$.*

Then M admits a direct sum decomposition $M_0 \oplus M_1 \oplus \cdots$ such that

$$f_1(M_0, r) = \cdots = f_{\mathrm{rank}(M_0)}(M_0, r) = r \qquad (r > r_0)$$

and for each $k > 0$, there is a constant $c_k > r_0$ such that

$$f_1(M_k, r) = \cdots = f_{\mathrm{rank}(M_k)}(M_k, r) = \max\{r, c_k\} \qquad (r > r_0).$$

In particular, for $j = i + 1, \dots, n$, $f_j(M, r) = r$ for $r > r_0$.

Proof. We induct on i. Suppose first that $i = 0$. In this case, we cannot have $f_1(M, r_1) > r_1$ for any $r_1 > r_0$, as then Proposition 3.6.3(d) would imply $f_1(M, r) > r_1$ for all $r \in (r_0, r_1)$ and hence $\liminf_{r \to r_0^+} f_1(M, r) \geq r_1$, violating condition (b). Hence for all $r > r_0$ and all j, we have $r = f_1(M, r) \geq f_j(M, r) \geq r$, proving the claim with $M = M_0$.

Suppose next that $i > 0$. Let c_1 be the constant value of $f_1(M, r)$ for $r \in (r_0, r_1)$. By Corollary 3.6.5, we have $f_1(M, r) = \max\{r, c_1\}$ for $r > r_0$. Let m be the largest value for which $f_1(M, r) = f_m(M, r)$ for r in some right neighborhood (r_0, r_1) of r_0. Split M as $M_1 \oplus M_2$ as per Proposition 3.6.7 so that M_1 accounts for the first m intrinsic subsidiary radii of M_ρ for $\rho > e^{-r_1}$. For $r \geq c_1$, for all j we have $r \leq f_j(M_1, r) \leq f_1(M_1, r) \leq f_1(M, r) = r$ and so $f_j(M_1, r) = r$. By Proposition 3.6.3(d), $F_{\mathrm{rank}(M_1)}(M_1, r)$ is convex; since it agrees with the constant function $\mathrm{rank}(M_1) c_1$ for $r \in (r_0, r_1)$ and for $r = c_1$, we must have $F_{\mathrm{rank}(M_1)}(M_1, r) = \mathrm{rank}(M_1) c_1$ for $r \in (r_0, c_1]$. Since in addition $f_j(M_1, r) \leq f_1(M_1, r) \leq f_1(M, r) = \max\{r, c_1\}$, we must have $f_j(M_1, r) = c_1$ for $r \in (r_0, c_1]$. Thus M_1 has all of the desired properties, so we may apply the induction hypothesis to M_2 to prove the claim. □

Proposition 3.6.9. *Suppose that $I = (\alpha, \beta)$ for some $\alpha, \beta > 0$ and that for some $i \in \{1, \dots, n-1\}$, the following conditions hold.*

(a) *The function $F_i(M, r)$ is affine for $r \in -\log I$.*
(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $r \in -\log I$.*

Then M admits a unique direct sum decomposition separating the first i intrinsic subsidiary radii of M_ρ for every $\rho \in I$.

Proof. See [25, Theorem 12.4.2]. □

3.7. **Decompositions over open annuli.** We now embark on a deeper analysis of differential modules over open annuli than is found in [25], concentrating on spectral decompositions and on properties of refined modules. For the latter, we incorporate some ideas of Xiao [46, 47].

Hypothesis 3.7.1. Throughout §3.7, continue to retain Hypothesis 3.6.1, but assume further that $n > 0$ and $I = (\alpha, \beta)$ for some $\alpha, \beta > 0$.

Definition 3.7.2. We say that M is *pure* if the functions $f_1(M, r), \dots, f_n(M, r)$ for $r \in -\log I$ are all equal to a single affine function. A *spectral decomposition* of M is a direct sum decomposition $M = \bigoplus_i M_i$ in which each summand M_i is pure and the values $f_1(M_i, r)$ are all distinct for each $r \in -\log I$. If such a decomposition exists, it specializes to the spectral decomposition of M_ρ for all $\rho \in I$; in particular, a spectral decomposition is unique if it exists.

Lemma 3.7.3. *Suppose that the following conditions hold.*

(a) *For $i = 1, \dots, n$, the function $f_i(M, r)$ is affine for $r \in -\log I$.*
(b) *For $i = 1, \dots, n-1$, either $f_i(M, r) = f_{i+1}(M, r)$ for $r \in -\log I$ or $f_i(M, r) > f_{i+1}(M, r)$ for $r \in -\log I$.*

Then M admits a spectral decomposition.

Proof. This is immediate from Proposition 3.6.9. □

Definition 3.7.4. Suppose that M admits a spectral decomposition. By the *Robba component* of M, we mean the summand M_1 in the spectral decomposition of M for which $f_1(M_1, r) = r$ for each $r \in -\log I$, or the zero submodule if no such summand exists.

Lemma 3.7.5. *Suppose that M admits a spectral decomposition. Let M_1 be the Robba component of M. Then the natural maps $H^i(M_1) \to H^i(M)$ are bijections for $i = 0, 1$.*

Proof. Let M_2 be the complementary summand of M_1 in the spectral decomposition of M. It is clear that $H^0((M_2)_\rho) = 0$ for $\rho \in I$, proving the desired bijectivity for $i = 0$. For $i = 1$, note that $f_1(M_2, r) > r$ for each $r \in -\log I$, so any extension $0 \to R_I \to N \to M_2 \to 0$ splits by Lemma 3.7.3. □

Lemma 3.7.6. *Suppose that M admits a spectral decomposition. Let M_1 be the Robba component of M. Assume either that $p = 0$ or that $p > 0$ and M_1 has p-adic non-Liouville exponents.*

(a) *Let M_2 be the maximal unipotent submodule of M_1. Then the natural maps $H^i(M_2) \to H^i(M)$ are bijections for $i = 0, 1$.*
(b) *The composition $H^0(M) \times H^1(M^\vee) \to H^1(R_I) \to K$ in which the first map is induced by the natural pairing $M \times M^\vee \to R_I$ and the second map is the residue map is a perfect pairing of finite-dimensional K-vector spaces.*
(c) *For any open subinterval J of I, the map*
$$H^i(M) \to H^i(M_J)$$
is a bijection for $i = 0, 1$.

Proof. To prove (a), we may replace M by M_1 using Lemma 3.7.5. In case $p = 0$, we may check the claim after replacing K by a finite extension K', since M may be viewed as a direct summand of $M \otimes_K K'$. After a suitable such extension, by Theorem 3.3.6 we may decompose $M_1 = M_2 \oplus M_3$ in such a way that M_3 becomes a successive extension of copies of M_λ for various $\lambda \in \mathfrak{o}_K \setminus \mathbb{Z}$. To see that $H^i(M_3) = 0$ for $i = 0, 1$, we may use the snake lemma to reduce to the case $M = M_\lambda$ for some $\lambda \in \mathfrak{o}_K \setminus \mathbb{Z}$. In this case, vanishing of H^0 follows from the nontriviality of M_λ, while vanishing of H^1 follows from Theorem 3.3.6 applied to an extension $0 \to R_I \to N \to M_\lambda \to 0$.

To prove (a) in case $p > 0$, apply Corollary 3.4.23 to decompose $M_1 = M_2 \oplus M_3$ where M_3 has an exponent containing no integer or p-adic Liouville number. On one hand, $H^0(M_3) = 0$ because otherwise Remark 3.4.14 would force M to have an exponent containing 0. On the other hand, $H^1(M_3) = 0$ because we may split any extension $0 \to R_I \to N \to M_3 \to 0$ using Remark 3.4.14 and Theorem 3.4.22.

To prove (b) and (c), we may use (a) to reduce to the case $M = M_2$. We may then use the snake lemma to reduce to the case $M = R_I$, for which both claims are easily verified. □

Lemma 3.7.7. *Suppose that M admits a spectral decomposition. Assume either that $p = 0$ or that $p > 0$ and the Robba component of M has p-adic non-Liouville exponent differences. Then for any $\rho \in I$, the map $H^0(M) \to H^0(M_\rho)$ is a bijection.*

Proof. In case $p = 0$, this is immediate from Lemma 3.7.6(a). In case $p > 0$, apply Corollary 3.4.24 to reduce to the case $M = M_\lambda$ for some $\lambda \in \mathbb{Z}_p$. The claim then holds because by [25, Proposition 9.5.3], $H^0(M_{\lambda,\rho}) = 0$ whenever $\lambda \notin \mathbb{Z}$. □

Definition 3.7.8. We say that M is *refined* if M is pure and moreover $f_1(M,r) > f_1(M^\vee \otimes M, r)$ for all $r \in -\log I$ (that is, M is pure and M_ρ is refined for all $\rho \in I$). If M_1, M_2 are refined, we say they are *equivalent* if $f_1(M_1^\vee \otimes M_2, r) < f_1(M_1,r), f_1(M_2,r)$ for all $r \in -\log I$. Note that if M_1 and M_2 are inequivalent, then by convexity (Proposition 3.6.3(e)) we must have $f_1(M_1^\vee \otimes M_2, r) = \max\{f_1(M_1,r), f_1(M_2,r)\}$ for all $r \in -\log I$.

A *refined decomposition* of M is a direct sum decomposition in which each summand M_i is either refined or satisfies the Robba condition, at most one summand satisfies the Robba condition, and any two distinct refined summands M_i, M_j are inequivalent. Such a decomposition specializes to a refined decomposition of M_ρ for each $\rho \in I$, and hence is unique if it exists.

It is easiest to obtain refined decompositions using the following construction of *test modules* (compare [47, Example 1.3.20]).

Definition 3.7.9. For any finite tamely ramified extension K' of K, any $\lambda \in K'$, any positive integer m not divisible by p, any positive integer e which is a power of p (which must be 1 if $p = 0$), and any integer h coprime to em, let $N_{\lambda,h,e,m}$ be the differential module over $(R_I \otimes_{K[t]} K'[t^{1/m}], \frac{d}{dt^{1/m}})$ on the generators $\mathbf{v}_1, \dots, \mathbf{v}_e$ given by

$$D(\mathbf{v}_1) = t^{-1/m}\mathbf{v}_2, \ \dots, \ D(\mathbf{v}_{e-1}) = t^{-1/m}\mathbf{v}_e, \ D(\mathbf{v}_e) = \lambda t^{-1/m+h/m}\mathbf{v}_1.$$

Lemma 3.7.10. *With notation as in Definition 3.7.9, for $\rho > 0$ we have*

$$\min\{\omega, IR((N_{\lambda,h,e,m})_\rho)\} = \min\{\omega, \omega\, |\lambda|^{-1/e}\, \rho^{-h/(em)}\}.$$

Proof. This is immediate from Proposition 2.2.6. □

Lemma 3.7.11. *Suppose that M is pure and $f_1(M,r) > r - \log\omega$ for $r \in -\log I$. Then for any $\rho \in I$, there exist a finite tamely ramified extension K' of K and a positive integer m not divisible by p such that $M_\rho \otimes_{K[t]} K'[t^{1/m}]$ admits a refined decomposition in which for each summand V, there exist a scalar $\lambda \in K'$, a positive integer e which is a power of p, and an integer h coprime to em such that $IR(M_\sigma) = IR((N_{\lambda,h,e,m})_\sigma)$ for σ in some neighborhood of ρ and $IR(V^\vee \otimes (N_{\lambda,h,e,m})_\rho) > IR(V)$.*

Proof. We imitate the proof of [25, Lemma 6.8.1]. Apply Corollary 2.1.6 to produce $\mathbf{v} \in M$ which is a cyclic vector in $M \otimes_{R_I} \mathrm{Frac}(R_I)$. Write $D^n(\mathbf{v}) = a_0\mathbf{v} + \cdots + a_{n-1}D^{n-1}(\mathbf{v})$ with $a_0, \dots, a_{n-1} \in \mathrm{Frac}(R_I)$. Factor the polynomial $P(T) = T^n - a_{n-1}T^{n-1} - \cdots - a_0$ over an algebraic closure of $\mathrm{Frac}(R_I)$ within an algebraic closure of F_ρ. For each root

α, we can find λ, h, e, m such that $\left|\alpha - m^{-1}\lambda^{1/e}t^{-1+h/(em)}\right|_\rho < |\alpha|_\rho$; by Corollary 2.2.7, $IR(M_\sigma) = IR((N_{\lambda,h,e,m})_\sigma)$ for σ in a neighborhood of ρ and one of the intrinsic subsidiary radii of $M_\rho^\vee \otimes (N_{\lambda,h,e,m})_\rho$ is greater than $IR(M_\rho)$. Apply Proposition 2.2.9 to construct a refined decomposition of $M_\rho \otimes_{F_\rho} E$ for some finite tamely ramified extension E of F_ρ; then each summand is equivalent to $(N_{\lambda,h,e,m})_\rho$ for some λ, h, e, m, and in particular is stable under $\mathrm{Gal}(E'/F')$ for $F' = F_\rho \otimes_{K[t]} K'[t^{1/m}]$ and E' a compositum of E, F', and $K(\mu_m)$. We thus obtain a refined decomposition of $M_\rho \otimes_{K[t]} K'[t^{1/m}]$ with the desired property. $\square$

Theorem 3.7.12. *Suppose that M is pure. Then there exist a finite tamely ramified extension K' of K and a positive integer m not divisible by p such that $M \otimes_{K[t]} K'[t^{1/m}]$ admits a refined decomposition.*

Proof. By virtue of the uniqueness of refined decompositions, we may work locally in a neighborhood of some $\rho \in (\alpha, \beta)$. Suppose first that $IR(M_\rho) < \omega$. To simplify notation, we may assume that the conclusion of Lemma 3.7.11 holds with $K' = K$ and $m = 1$, so that M_ρ admits a refined decomposition. In addition, for each summand V in the refined decomposition of M_ρ, we can find a differential module N over R_I such that $IR(V^\vee \otimes N_\rho) > IR(V)$. By continuity (Proposition 3.6.3(a)), for σ in a neighborhood of ρ, $M_\sigma^\vee \otimes N_\sigma$ has an intrinsic subsidiary radius strictly greater than $IR(M_\sigma) = IR(N_\sigma)$. Apply Proposition 3.6.9 to $N^\vee \otimes M$ to pull off a summand corresponding to the intrinsic subsidiary radii of $N_\rho^\vee \otimes M_\rho$ less than $IR(V)$, then tensor with N and project the decomposition from $N \otimes N^\vee \otimes M$ to M. Repeating this process gives the desired decomposition.

Suppose next that $p > 0$ and $IR(M_\rho) = \omega$. Let M' be the global Frobenius descendant of M (Definition 3.5.2). By Proposition 2.3.5, $IR(\varphi_* M_\rho) = \omega^p$, so we may apply the previous paragraph to exhibit a finite tamely ramified extension K' of K and a positive integer m not divisible by p such that $M' \otimes_{K[t^p]} K'[t^{p/m}]$ admits a refined decomposition. To simplify notation, we may assume that $K' = K$ and $m = 1$, i.e., that M' itself admits a refined decomposition. In particular, $\varphi_* M_\rho$ admits a refined decomposition. By Remark 2.3.10, if we group summands of $\varphi_* M$ into $\mathbb{Z}/p\mathbb{Z}$-orbits, the resulting decomposition descends to a decomposition specializing to a refined decomposition of M.

Suppose finally that $p > 0$ and $IR(M_\rho) > \omega$. Using Frobenius antecedents (Proposition 3.5.3), we may reduce to one of the previous cases. $\square$

Theorem 3.7.13. *Suppose that either:*

(a) *M is refined and* $\mathrm{rank}(M)$ *is not divisible by p; or*

(b) *$p > 0$ and M is of cyclic type.*

Then the slopes of $f_1(M, r)$ are in $\mathbb{Z}$.

Proof. Using Proposition 3.6.3(a), $f_1(M, r)$ is piecewise affine. It thus suffices to compute its slope on a closed subinterval J of I on which $f_1(M, r)$ is affine. We may assume that this slope is not equal to 0 or 1, as otherwise there is nothing left to check.

Suppose first that we are in case (a) with $p = 0$. Choose a generator $\mathbf{v}$ of $\wedge^n M_J$, define $c \in R_J$ by the formula $D(\mathbf{v}) = c\mathbf{v}$, and let N be the differential module over R_J on a single generator $\mathbf{v}$ given by $D(\mathbf{w}) = (c/n)\mathbf{v}$. We then have $N^{\otimes n} \cong \wedge^n M$ and so $f_1(N^\vee \otimes M, r) < f_1(M, r)$ for $r \in I$ by [25, Proposition 6.8.4]. In particular, in some range we have $f_1(M, r) = f_1(N, r)$, whereas $f_1(N, r)$ has integer slopes by Proposition 3.6.3(b). This proves the claim in this case.

Suppose next that we are in case (a) with $p > 0$. Since we assumed that the slope of $f_1(M, r)$ is neither 0 nor 1, we may shrink J to ensure that $f_1(M, r) \neq r - p^{-j} \log \omega$ for all $r \in J$ and all nonnegative integers j. We may then use Frobenius antecedents (Proposition 3.5.3) to reduce to the case where $f_1(M, r) > r - \log \omega$ for all $r \in J$, and then argue as in (a).

Suppose finally that we are in case (b). We may again assume that $f_1(M, r) > r - \log \omega$ for all $r \in J$; we may also assume that K is algebraically closed. Pick any $r_0 \in J$ and apply Lemma 3.7.11 to construct λ, h, e, m for which $IR(M_\rho^\vee \otimes (N_{\lambda,h,e,m})_\rho) > IR(M_\rho)$ for $\rho = e^{-r_0}$; by continuity (Proposition 3.6.3(a)), the same inequality holds for ρ in a neighborhood of e^{-r_0}. For $\mu \in 1 + \mathfrak{m}_K$, apply Corollary 2.2.7(a) to $\mu^* N^\vee_{\lambda,h,e,m} \otimes N_{\lambda,h,e,m}$; it implies that there exists $a > 0$ for which $f_1(\mu^* M^\vee \otimes M, r) = f_1(M, r) + a \log |\mu - 1|$ for $|\mu - 1|$ sufficiently close to 1 and r sufficiently close to r_0. By Lemma 3.2.6, there exists a rank 1 submodule Q_μ of $\mu^* M^\vee \otimes M$. Since $\mu^* M^\vee \otimes M$ is of cyclic type, we have $f_1(Q_\mu, r) = f_1(\mu^* M^\vee \otimes M, r) = f_1(M, r) + a \log |\mu - 1|$ for suitable μ, r. Since $f_1(Q_\mu, r)$ has integer slopes by Proposition 3.6.3(b) again, so then does M in a neighborhood of r_0; this proves the claim in this case. □

Remark 3.7.14. Theorem 3.7.13(b) is new to this paper. It was known previously that if $p > 0$, M is of cyclic type, and $\mathrm{End}(M)$ has p-adic non-Liouville exponent differences, then M is a successive extension of differential modules of rank 1 over R_I; namely, this is an easy consequence of Corollary 3.4.24. This result figures in the proofs of the p-adic local monodromy theorem given by André [1] and Mebkhout [33]; see Remark 3.8.19.

The following refinement of Lemma 3.7.11 will be used in the study of solvable modules in §3.8.

Lemma 3.7.15. *Choose γ, δ with $\alpha < \gamma < \delta < \beta$. Suppose that $p > 0$, K is algebraically closed, M is refined, and there exists a nonnegative integer b such that $IR(M_\rho) = (\alpha/\rho)^b < \omega$ for $\rho \in [\gamma, \delta]$. Then there exists a differential module N over R_I which is free of rank 1 with $IR(N_\rho) = (\alpha/\rho)^b$ for $\rho \in (\alpha, \delta]$ and $IR((N^\vee \otimes M)_\rho) < IR(M_\rho)$ for $\rho \in [\gamma, \delta]$.*

Proof. We may rescale to reduce to the case $\rho = \alpha = 1$. Using Lemma 3.7.11, we may replace M with $N_{\lambda,h,e,m}$; note that the fact that $b \in \mathbb{Z}$ forces $e = m = 1$. After making the substitution $t \mapsto t^{-1}$, we may perform the construction from the proof of [25, Theorem 12.7.2] to obtain the desired N. □

3.8. **Solvable modules.** We continue in the vein of [25], next treating differential modules over rings of convergent power series on an open annulus which are *solvable at a boundary*. This gives a uniform statement of the classical Turrittin-Levelt-Hukuhara decomposition as well as a strong p-adic analogue.

Note that for differential modules on an open annulus, one can equally well discuss solvability at the inner boundary or the outer boundary. In [25] and other literature, it is typical to consider outer boundaries because one has in mind the boundary of a residue disc. However, in this paper we will mostly need to consider inner boundaries (see §4.4), so we will set notation to address that case.

Hypothesis 3.8.1. Throughout §3.8, fix $\alpha > 0$ and put

$$\mathcal{R}_\alpha = \bigcup_{\beta > \alpha} R_{(\alpha,\beta)},$$

viewed as a differential ring for the derivation $d = \frac{d}{dt}$. Let M be a differential module over $\mathcal{R}_\alpha$ which is *solvable at α* in the sense of Definition 3.8.3 below.

Convention 3.8.2. The functions $f_i(M, r)$ and $F_i(M, r)$ are not well-defined for any particular $r < -\log \alpha$; however, the germs of these functions in left neighborhoods of $-\log \alpha$ may be interpreted unambiguously. We will use these germs frequently in what follows.

Definition 3.8.3. The module M is *solvable at α* if

$$\lim_{r \to (-\log \alpha)^-} f_1(M, r) = -\log \alpha.$$

By Proposition 3.6.3 plus an extra argument (see [25, Lemma 12.6.2]), this implies that there exist nonnegative rational numbers $b_1(M) \geq \cdots \geq b_n(M)$ such that at the level of germs, we have

$$f_i(M,r) = r + b_i(M)(-\log \alpha - r) \qquad (i = 1, \dots, n). \tag{3.8.3.1}$$

Definition 3.8.4. We say M satisfies the *Robba condition* if $b_1(M) = 0$. We say M is *refined* if $b_1(M) > b_1(\mathrm{End}(M))$. We say M is of *cyclic type* if $b_1(\mathrm{End}(M)) = 0$.

Lemma 3.8.5. *Suppose either that:*

(a) $p = 0$ *and* M *is refined; or*
(b) $p > 0$, K *is algebraically closed,* M *is refined, and* $\dim(M)$ *is not divisible by* p*; or*
(c) $p > 0$, K *is algebraically closed,* M *is of cyclic type, and* $b_1(M) > 0$.

Then there exists a differential module N *over* $\mathcal{R}_\alpha$ *which is free of rank* 1*, is solvable at* α*, and satisfies* $b_1(N^\vee \otimes M) < b_1(M)$.

Proof. Realize M as a refined differential module over $R_{(\alpha,\beta)}$ for some $\beta > \alpha$. By Theorem 3.7.13, $b_1(M)$ is a positive integer. We may thus imitate the proof of [25, Theorem 12.7.2] as follows.

In case (a), we may apply Lemma 3.7.11 to construct $N_{\lambda,h,e,m}$ with $IR((N^\vee_{\lambda,h,e,m} \otimes M)_\rho) < IR(M_\rho)$ for ρ in some interval; because $b_1(M) \in \mathbb{Z}$, we are forced to take $e = m = 1$. By Lemma 3.7.10, $N_{\lambda,h,1,1}$ is solvable at α. It remains to check that we may take λ in K, not just in a finite extension of K; for this, we argue as in Proposition 2.2.11. Put $n = \mathrm{rank}(M)$. Choose a generator $\mathbf{v}$ of the restriction of $\wedge^n M$ to R_I for some closed interval I, and write $D(\mathbf{v}) = a\mathbf{v}$ with $a \in R_I$. Let M' be the differential module over R_I on the single generator $\mathbf{w}$ with $D(\mathbf{w}) = (a/n)\mathbf{w}$; then $(M')^{\otimes n}$ is isomorphic to the restriction of $\wedge^n M$ to R_I. It follows that $\left|a/n - \lambda t^{h-1}\right|_\rho < |a/n|_\rho = \left|\lambda t^{h-1}\right|_\rho$ for $\rho \in I$, so there must exist $\lambda' \in K$ with $|\lambda - \lambda'| < |\lambda| = |\lambda'|$. We may thus replace $N_{\lambda,h,1,1}$ with $N_{\lambda',h,1,1}$ without affecting the preceding arguments.

In cases (b) and (c), by taking global Frobenius antecedents (Proposition 3.5.3) as needed, we can ensure that there exist γ, δ with $\alpha < \gamma < \delta < \beta$ such that $IR(M_\rho) > \omega$ for $\rho \in [\gamma, \delta]$. By Lemma 3.7.15, we obtain the desired module N. $\square$

Corollary 3.8.6. *Suppose either that:*

(a) $p = 0$ *and* M *is indecomposable and refined; or*
(b) $p = 0$ *and* M *is of cyclic type; or*
(c) $p > 0$, K *is algebraically closed,* M *is indecomposable and refined, and* $\dim(M)$ *is not divisible by* p*; or*

(d) *$p > 0$, K is algebraically closed, M is of cyclic type, and $b_1(M) > 0$.*

Then there exists a factorization $M \cong N \otimes P$ in which N is free of rank 1 and $b_1(P) = 0$. In particular, M is of cyclic type.

Proof. This follows by repeated application of Lemma 3.8.5. Note that since $b_1(M) \in \mathbb{Z}$ by Theorem 3.7.13, only finitely many iterations are needed before $b_1(M)$ is reduced to 0. □

When $p = 0$, the structure of solvable modules is relatively simple.

Theorem 3.8.7. *Assume $p = 0$. Then there exist a finite extension K' of K and a positive integer m such that $M \otimes_{K[t]} K'[t^{1/m}]$ admits a direct sum decomposition in which each summand is of cyclic type.*

Proof. This follows from Theorem 3.7.12 and Corollary 3.8.6. □

Remark 3.8.8. By taking $K = \mathbb{C}$ with the trivial norm, we may deduce from Theorem 3.8.7 the usual Turritin-Levelt-Hukuhara decomposition theorem for differential modules over $\mathbb{C}((t))$ [25, Theorem 7.5.1].

Definition 3.8.9. Put $F = \mathrm{Frac}(\mathcal{R}_\alpha)$. Let $[M]$ denote the Tannakian subcategory generated by M within the category of differential modules over $\mathcal{R}_\alpha$, equipped with the fibre functor ω taking each $N \in [M]$ to the F-vector space $N \otimes_{\mathcal{R}_\alpha} F$. Note that the objects of $[M]$ are all solvable at α.

Let $G(M)$ be the automorphism group of ω. For $r \geq 0$, let $G^r(M)$ denote the subgroup of $G(M)$ which acts trivially on $\omega(N)$ for each nonzero $N \in [M]$ for which $b_1(N) < r$. Also put $G^{r+}(M) = \bigcup_{s>r} G^s(M)$.

Remark 3.8.10. As in Remark 2.3.19, we may use Theorem 3.8.7 to deduce that when $p = 0$, the group $G^{0+}(M)$ is a torus. The structure of $G^{0+}(M)$ in case $p > 0$ will be clarified by Theorem 3.8.12 below; this will imply that for any p and any $r \geq 0$, $G^{r+}(M)$ equals the subgroup of $G(M)$ which acts trivially on $\omega(N)$ for each nonzero $N \in [M]$ for which $b_1(N) \leq r$.

Lemma 3.8.11. *If $p > 0$ and M is of cyclic type, then there exists a nonnegative integer h such that $b_1(M^{\otimes p^h}) = 0$.*

Proof. If $b_1(M) > 0$, then by Proposition 2.3.13, we have $b_1(M^{\otimes p}) < b_1(M)$. Since $b_1(M)$ and $b_1(M^{\otimes p})$ are nonnegative integers by Theorem 3.7.13, this proves the claim. □

Theorem 3.8.12. *If $p > 0$, then $G^{0+}(M)$ is a finite p-group.*

Proof. This follows from Proposition 1.1.2 using Remark 1.1.3 as follows. Replace the category of differential modules over $\mathcal{R}_\alpha$ with the direct limit of the categories of differential modules over $\mathcal{R}_\alpha \otimes_{K[t]} K'[t^{1/m}]$ over all finite extensions K' of K and all positive integers m not divisible by p; this does not change the groups $G^r(M)$ except for a base extension. We may then deduce conditions (i), (ii), (iii) of Remark 1.1.3 using Theorem 3.7.12, Proposition 2.3.13, Lemma 3.8.11, respectively. □

Corollary 3.8.13. *There exist a finite extension K' of K and a positive integer m such that for all nonnegative integers g, h, $(M^\vee)^{\otimes g} \otimes M^{\otimes h} \otimes_{K[t]} K'[t^{1/m}]$ admits a refined decomposition.*

Proof. This is apparent from Theorem 3.8.7 if $p = 0$. If $p > 0$, for each pair (g, h) we may choose a suitable m by Theorem 3.7.12, so we need only check that m may be chosen uniformly. But this follows from Theorem 3.8.12: it is enough to list each of the finitely many isomorphism classes of irreducible representations τ of $G^{0+}(M)$ and, for each τ, ensure that m works for one pair g, h such that τ appears in $(M^\vee)^{\otimes g} \otimes M^{\otimes h}$. □

Corollary 3.8.14. *If $p > 0$ and $b_1(M) > 0$, then there exist a finite extension K' of K, a positive integer m and an object $N \in [M \otimes_{K[t]} K'[t^{1/m}]]$ of cyclic type such that $b_1(N) > 0$ but $b_1(N^{\otimes p}) = 0$.*

Proof. This follows from Theorem 3.8.12 and the fact that any nontrivial finite p-group admits a normal subgroup of index p. □

Lemma 3.8.15. *Suppose that $p > 0$, K contains a primitive p-th root of unity, M is free of rank 1, and $b_1(M^{\otimes p}) = 0$. Then there exists another differential module N over $\mathcal{R}_\alpha$ which is solvable on α, is free on a single generator $\mathbf{v}$ such that $D(\mathbf{v}) = P'(t)$ for some $P(t) \in K[t]$ with $|P(t)|_\alpha = \omega$, and satisfies $b_1(N^\vee \otimes M) = 0$.*

Proof. This follows from [25, Theorem 17.1.6, Remark 17.1.7]. □

Definition 3.8.16. Let $\mathcal{R}_\alpha^{\mathrm{bd}}$ be the subring of $\mathcal{R}_\alpha$ consisting of germs of bounded analytic functions. This ring is henselian but not complete for the α-Gauss norm; let $\mathcal{R}_\alpha^{\mathrm{int}}$ denote the valuation subring.

If S is a connected finite étale cover, it makes sense to impose the Robba condition on $M \otimes_{\mathcal{R}_\alpha^{\mathrm{int}}} S$ provided that S can be identified with a ring of the form $\mathcal{R}_\alpha^{\mathrm{int}}$ in a suitable power series coordinate; the resulting condition will not depend on the choice of this identification. Such an identification can always be made if κ_K is algebraically closed.

Theorem 3.8.17. *If $p > 0$, then there exists a connected finite étale cover S of $\mathcal{R}_\alpha^{\mathrm{int}}$ such that $M \otimes_{\mathcal{R}_\alpha^{\mathrm{int}}} S$ satisfies the Robba condition in the sense of Definition 3.8.16.*

Proof. Since $G^{0+}(M)$ is finite by Theorem 3.8.12 and is trivial if and only if M satisfies the Robba condition, it suffices to produce a cover that decreases $G^{0+}(M)$. This may be achieved as follows. We may assume from the outset that K contains an element π with $\pi^{p-1} = -p$; this also forces K to contain a primitive p-th root of unity. Pick out an object $N \in [M \otimes_{K[t]} K'[t^{1/m}]]$ for some K', m as in Corollary 3.8.14. Apply Corollary 3.8.6 to produce a free rank 1 object $N' \in [M \otimes_{K[t]} K'[t^{1/m}]]$ for some K', m such that $N^\vee \otimes N'$ satisfies the Robba condition. By Lemma 3.8.15, we may choose N' to be free on one generator $\mathbf{v}$ satisfying $D(\mathbf{v}) = P'(t)$ for some $P \in K[t]$ with $|P(t)|_\alpha = \omega$. We may then trivialize N' by extending scalars from $\mathcal{R}_\alpha^{\mathrm{int}}$ to $\mathcal{R}_\alpha^{\mathrm{int}}[z]/(z^p - z - \pi^{-1}P(t))$ and recalling that the power series $\exp(\pi(z^p - z))$ in z has radius of convergence strictly greater than 1 (see for example [25, Example 9.9.3]). □

Corollary 3.8.18. *Assume that $p > 0$, κ_K is algebraically closed, and $\alpha = 1$.*

(a) *There is a unique minimal choice of S satisfying the conclusion of Theorem 3.8.17.*
(b) *The residue field of S is a finite Galois extension of $\kappa_K((t))$ whose highest ramification break is equal to $b_1(M)$.*

Proof. This follows from Theorem 3.8.17 as in the proof of [21, Theorem 5.23] (see also [25, Theorem 19.4.1]). □

Remark 3.8.19. Theorem 3.8.17 includes a result variously known as the *p-adic Turrittin theorem* and the *p-adic local monodromy theorem.* This result, due to André [1], Mebkhout [33], and the author [20], assumes the existence of a *Frobenius structure* on M (see [25, Chapter 17]); in addition, K must be discretely valued and β must equal 1.

The methods of André and Mebkhout can be used to derive Theorem 3.8.17 also in the case where all of the objects in $[M]$ have p-adic non-Liouville exponent differences. In these arguments, the non-Liouville condition is needed to ensure that irreducible objects satisfying the Robba condition are all of rank 1. The proof of Theorem 3.8.17 provides a workaround in cases where advance information about exponents is not available.

4. Berkovich discs

We are at last ready to shift language and perspective towards Berkovich's nonarchimedean analytic spaces. In this section, we introduce the topological spaces which play the role of discs in Berkovich's theory, and consider radii of convergence of local horizontal sections of differential modules on such spaces. This draws heavily on the results of §3, but some additional maneuvering is needed. In addition, the behavior of differential modules around points of type 4 requires some extra work.

4.1. **Underlying topological spaces.**

We begin by defining the Gel'fand spectrum of a Banach ring. For now, we just consider the resulting topological space; we postpone discussion of the analytic space structure to §5.

Definition 4.1.1. For R a ring equipped with a submultiplicative norm (e.g., a commutative Banach algebra over K), the *Gel'fand spectrum* $\mathcal{M}(R)$ is defined as the set of bounded (by the given norm) multiplicative seminorms on R, topologized as a subset of the product $\mathbb{R}^R$. Note that $\mathcal{M}(R)$ may also be viewed as a closed subset of a product of bounded closed intervals, and hence is compact; it is also nonempty provided that $R \neq 0$ [8, Theorem 1.2.1]. For $x \in \mathcal{M}(R)$, let $\mathcal{H}(x)$ denote the completion of $\mathrm{Frac}(R/\ker(x))$ for the multiplicative norm induced by x.

Remark 4.1.2. Any bounded homomorphism $R \to S$ of commutative Banach algebras over K defines a continuous restriction map $\mathcal{M}(S) \to \mathcal{M}(R)$. If this map is surjective, then it is a quotient map because the source and target are compact (see for instance [30, Remark 2.3.15(a)]).

For example, suppose that R is a commutative Banach algebra over K and that K' is a complete field extension of K. Then the completed tensor product $R' = R \widehat{\otimes}_K K'$ is a Banach algebra over K', and the restriction map $\mathcal{M}(R') \to \mathcal{M}(R)$ is always surjective (see for instance [30, Lemma 2.3.13]).

In the previous paragraph, if K' is the completion of an algebraic Galois extension of K (such as $\mathbb{C}$), we can say more: not only is the restriction map $\mathcal{M}(R') \to \mathcal{M}(R)$ surjective, but the group of continuous automorphisms of K' over K acts transitively on the fibres of the restriction map. See [8, Corollary 1.3.6].

Definition 4.1.3. Let R be a commutative Banach algebra over K, and put $R' = R \widehat{\otimes}_K \mathbb{C}$. For $x \in \mathcal{M}(R)$, choose any lift $\tilde{x} \in \mathcal{M}(R')$ of x,

and define the *signature* of x as the triple

$$\left(\dim(\ker(\tilde{x})), \operatorname{rank}(|\mathcal{H}(x)^\times|/|K^\times|), \operatorname{trdeg}(\kappa_{\mathcal{H}(x)}/\kappa_K)\right).$$

Note that one can have $\dim(\ker(\tilde{x})) > \dim(\ker(x))$.

4.2. **Discs.** We now specialize the previous discussion to rings of convergent power series on discs. Due to the increasing prevalence of such rings and their associated Gel'fand spectra in various branches of mathematics, numerous expositions of this material can be found in the literature; among these, perhaps the most comprehensive is the book of Baker and Rumely [3, Chapter 1]. However, that treatment assumes that the ground field K is algebraically closed, which we prefer not to do here; to avoid imposing this condition, we refer also to [29, §2].

Definition 4.2.1. For $\beta > 0$, the space $\mathcal{M}(R_{[0,\beta]})$ is called the *Berkovich closed disc of radius* β with coordinate t over K, and also denoted $\mathbb{D}_{\beta,K}$. For $z \in \mathbb{C}$ with $|z| \leq \beta$ and $\rho \in [0,\beta]$, the restriction to $R_I \cong K\langle t/\beta \rangle$ of the ρ-Gauss norm on $\mathbb{C}\langle (t-z)/\beta \rangle$ defines a point $\zeta_{z,\rho} \in \mathbb{D}_{\beta,K}$; the point $\zeta_{0,\beta}$ is called the *Gauss point* of $\mathbb{D}_{\beta,K}$. For $\beta' > \beta$, the natural map $R_{[0,\beta']} \to R_{[0,\beta]}$ induces an inclusion $\mathbb{D}_{\beta,K} \to \mathbb{D}_{\beta',K}$; the direct limit of the $\mathbb{D}_{\beta,K}$ along these maps is called the *Berkovich affine line* over K.

Lemma 4.2.2. *The restriction map $\mathbb{D}_{\beta,\mathbb{C}} \to \mathbb{D}_{\beta,K}$ identifies $\mathbb{D}_{\beta,K}$ with the quotient of $\mathbb{D}_{\beta,\mathbb{C}}$ by the action of the group of continuous automorphisms of $\mathbb{C}$ over K.*

Proof. See [8, Proposition 1.3.5]. □

Proposition 4.2.3. *For $\beta > 0$, $x \in \mathbb{D}_{\beta,K}$ and $\rho \in [0,\beta]$, define*

$$H(x,\rho)(f) = \max\left\{\rho^i x\left(\frac{1}{i!}\frac{d^i}{dt^i}(f)\right) : i = 0,1,\dots\right\} \tag{4.2.3.1}$$

with the interpretation $0^0 = 1$.

(a) *The formula (4.2.3.1) defines a continuous map*
$$H : \mathbb{D}_{\beta,K} \times [0,\beta] \to \mathbb{D}_{\beta,K}.$$
(b) *For $x \in \mathbb{D}_{\beta,K}$, $H(x,0) = x$ and $H(x,\beta) = \zeta_{0,\beta}$.*
(c) *For $x \in \mathbb{D}_{\beta,K}$ and $\rho,\sigma \in [0,\beta]$,*
$$H(H(x,\rho),\sigma) = H(x,\max\{\rho,\sigma\}).$$
(d) *For $z \in \mathbb{C}$ with $|z| \leq \beta$ and $\rho \in [0,\beta]$, $H(\zeta_{z,0},\rho) = \zeta_{z,\rho}$.*
(e) *For $x,y \in \mathbb{D}_{\beta,K}$, y dominates x (that is, $y(f) \geq x(f)$ for all $f \in R_I$) if and only if $y = H(x,\rho)$ for some $\rho \in [0,\beta]$.*

Proof. Ssee [8, Remark 6.1.3(ii)] or [29, Lemma 2.3] for (a)-(d) and [29, Theorem 2.11] for (e). □

Definition 4.2.4. For $\beta > 0$ and $x \in \mathbb{D}_{\beta,K}$, define the *diameter* of x, denoted $\rho(x)$, to be the maximum $\rho \in [0,\beta]$ for which $H(x,\rho) = x$. Beware that the diameter is stable under base extension from K to $\mathbb{C}$ (see Proposition 4.2.6), but not under general base extensions (see Remark 4.2.5). It is also stable under increasing β.

Remark 4.2.5. For $\beta > 0$ and $x \in \mathbb{D}_{\beta,K}$, let $t_x \in \mathcal{H}(x)$ be the image of t under the natural map $R_{[0,\beta]} \to \mathcal{H}(x)$. We may then realize x as the restriction of the seminorm $\zeta_{t_x,0} \in \mathcal{M}(R_{I,\mathcal{H}(x)})$ of radius 0.

In terms of the intrinsic radius function, Berkovich's classification of points of $\mathcal{M}(R_I)$ reads as follows.

Proposition 4.2.6. *For $\beta > 0$, every point of $\mathbb{D}_{\beta,K}$ is of exactly one of the following types (called* types 1,2,3,4 *hereafter).*

1. *Points of signature $(1,0,0)$. These are the points of the form $\zeta_{z,0}$ for some $z \in \mathbb{C}$. The diameter of such a point is 0.*
2. *Points of signature $(0,1,0)$. These are the points of the form $\zeta_{z,\rho}$ for some $z \in \mathbb{C}$ and some $\rho \in (0,\beta] \cap |\mathbb{C}^\times|$. The diameter of such a point is $\rho > 0$.*
3. *Points of signature $(0,0,1)$. These are the points of the form $\zeta_{z,\rho}$ for some $z \in \mathbb{C}$ and some $\rho \in (0,\beta] \setminus |\mathbb{C}^\times|$. The diameter of such a point is $\rho > 0$.*
4. *Points of signature $(0,0,0)$. The diameter of such a point x is the infimum of those values of ρ for which the seminorm x is dominated by some $\zeta_{z,\rho}$; it belongs to the interval $(0,\beta)$.*

Moreover, the points that are minimal under domination are precisely those of types 1 and 4.

Proof. For $K = \mathbb{C}$, see [8, 1.4.4]. For the general case, see [29, Theorem 2.26]. □

This can be used to recover a version of the Zariski-Abhyankar inequality. For a more traditional variant, see for instance [45, Théorème 9.2].

Corollary 4.2.7. *Let R be the completion of $K[T_1,\dots,T_d]$ for the Gauss norm. Then the signature of each point in $\mathcal{M}(R)$ consists of three nonnegative integers whose sum is at most d.*

Proof. Choose $x \in \mathcal{M}(R)$ and let x_i be the restriction of R to the completion of $K[T_1,\dots,T_i]$. Then the difference between the signatures of x_{i+1} and x_i is itself the signature of a point in $\mathbb{D}_{1,\mathcal{H}(x_i)}$. The claim thus follows from Proposition 4.2.6. □

We next make the topology of $\mathbb{D}_{\beta,K}$ more explicit.

Definition 4.2.8. For $\beta > 0$ and $x \in \mathbb{D}_{\beta,K}$, the *root path* of x is the subspace $\{H(x,\rho) : \rho \in [0,\beta]\}$ of $\mathbb{D}_{\beta,K}$. It is homeomorphic to the interval $[\rho(x),\beta]$ via the map $H(x,\cdot)$.

A *rooted skeleton* in $\mathbb{D}_{\beta,K}$ is a subset of the form

$$\bigcup_{i=1}^{m} \{H(x_i,\rho) : \rho \in [\rho(x_i),\beta]\}$$

for some nonempty finite subset $\{x_1,\dots,x_m\} \subseteq \mathbb{D}_{\beta,K}$; we sometimes say that this skeleton is *generated* by $x_1,\dots,x_m$. A *strict rooted skeleton* is a rooted skeleton generated by a set of points of type 2.

For any rooted skeleton S of $\mathbb{D}_{\beta,K}$, define the map $\pi_S : \mathbb{D}_{\beta,K} \to S$ taking each $x \in \mathbb{D}_{\beta,K}$ to $H(x,\rho)$ for ρ the least value in $[0,\beta]$ for which $H(x,\rho) \in S$. By Proposition 4.2.3, π_S is a deformation retract.

Proposition 4.2.9. *Form the inverse system consisting of the rooted skeleta of $\mathbb{D}_{\beta,K}$ with morphisms given as follows: for every pair of rooted skeleta S, S' with $S \subseteq S'$, include a morphism $S' \to S$ given by the restriction of π_S. Define a map from $\mathbb{D}_{\beta,K}$ to this inverse system whose projection onto S is given by π_S. Then this map is a homeomorphism of topological spaces.*

Proof. The map is injective because every pair of points can be found in some rooted skeleton. The map is surjective because $\mathbb{D}_{\beta,K}$ is compact and surjects onto each rooted skeleton. The map is a homeomorphism because any continuous bijection from a quasicompact space to a Hausdorff space is a homeomorphism. (See also [3, Proposition 1.13] for an alternate treatment in case K is algebraically closed and $\beta = 1$.) □

Remark 4.2.10. Proposition 4.2.9 is a special case of the general phenomenon that Berkovich analytic spaces can be described as inverse limits of tropical spaces (see for example [34]). For Berkovich curves, this inverse limit presentation is also closely related to semistable models; we will return to this point in §5.

Definition 4.2.11. For $x \in \mathbb{D}_{\beta,K}$, a *branch* of $\mathbb{D}_{\beta,K}$ at x is a path-connected component of $\mathbb{D}_{\beta,K} \setminus \{x\}$. If x is not the Gauss point, then there is a branch containing the Gauss point, called the *upper branch* of $\mathbb{D}_{\beta,K}$ at x. By Proposition 4.2.9, additional branches (called *lower branches*) exist according to the type of x as follows.

1. No lower branches.
2. Infinitely many lower branches.
3. Exactly one lower branch.

4. No lower branches.

For S a rooted skeleton of $\mathbb{D}_{\beta,K}$ and $x \in S$, a *branch* of S at x is a branch of X at x meeting S. There are only finitely many such branches at any x.

Definition 4.2.12. Let S be a rooted skeleton of $\mathbb{D}_{\beta,K}$. By a *subdivision* of S, we will mean a graph (in the combinatorial sense) with underlying topological space S.

We equip S with the piecewise linear structure characterized as follows: a function $f : S \to \mathbb{R}$ is piecewise affine (with integral slopes) if and only if for each $x \in S$, the function $r \mapsto f(H(x, e^{-r}))$ is piecewise affine (with integral slopes) and constant for r sufficiently large. Then for any piecewise affine function $f : S \to \mathbb{R}$, there exists a subdivision of S such the restriction of f to any edge of the subdivision is affine. We call such a subdivision a *controlling graph* of f.

It is meaningful to refer to the *slope* of a piecewise affine function $f : S \to \mathbb{R}$ along a branch of S at a point x. Explicitly, the slope along the upper branch is the left slope of $r \mapsto f(H(x, e^{-r}))$ at $r_0 = -\log \rho(x)$ (or 0 in case $\rho(x) = 0$), while the slope along the lower branch containing $y \in S$ is the right slope of $r \mapsto f(H(y, e^{-r}))$ at r_0.

Definition 4.2.13. By the *Berkovich open unit disc of radius β over K*, denoted $\mathbb{D}^{\circ}_{\beta,K}$, we will mean the branch of $\mathbb{D}_{\beta,K}$ at the Gauss point containing $\zeta_{0,0}$.

4.3. **Radii of convergence.** We now define the radii of optimal convergence for differential modules on discs, following Baldassarri [5].

Hypothesis 4.3.1. Throughout §4.3, fix $\beta > 0$ and (except in Definition 4.3.10 and Lemma 4.3.11) let M be a differential module of rank $n \geq 0$ over $R_{[0,\beta]}$.

Definition 4.3.2. For $x \in \mathbb{D}_{\beta,K}$, put

$$M_{x,0} = M \otimes_{R_{[0,\beta]}} \mathcal{H}(x)[\![t - t_x]\!],$$
$$M_{x,\rho} = M \otimes_{R_{[0,\beta]}} \mathcal{H}(x)\langle (t - t_x)/\rho \rangle \quad (\rho \in (0, \beta);$$

these can be viewed as differential modules as well. By a standard argument (see for instance [25, Theorem 7.2.1]), the natural map

$$M^{D=0}_{x,0} \otimes_{\mathcal{H}(x)} \mathcal{H}(x)[\![t - t_x]\!] \to M_{x,0}$$

is an isomorphism. Define the sequence $s_i(M, x)$ of *radii of optimal convergence* of M at x as follows: for $i = 1, \dots, n$, put

$$s_i(M, x) = \sup\{\rho \in [0, \beta) : \dim_{\mathcal{H}(x)}(M^{D=0}_{x,0} \cap M_{x,\rho}) \geq n - i + 1\}.$$

In other words, $s_i(M,x)$ is the radius of the maximal open disc around t_x on which there exist $n-i+1$ linearly independent horizontal sections of M. For $M \neq 0$, we refer to $s_1(M,x)$ also as the *radius of convergence* of M at x.

Lemma 4.3.3. *Let K' be an analytic field containing K, and suppose $y \in \mathbb{D}_{\beta,K'}$ restricts to $x \in \mathbb{D}_{\beta,K}$. Then*

$$s_i(M,x) = s_i(M \otimes_{R_{[0,\beta],K}} R_{[0,\beta],K'}, y) \qquad (i = 1, \dots, n).$$

Proof. By replacing K with $\mathcal{H}(x)$, we may reduce to the case $x = \zeta_{0,0}$. The claim then comes down to the fact that formation of the kernel of the bounded K-linear endomorphism of the Banach space $M \otimes_{R_{[0,\beta]}} R_{[0,\rho]}$ commutes with formation of the completed tensor product over K with K'. See [30, Lemma 2.2.10(b)]. □

Remark 4.3.4. The intuition behind Definition 4.3.2 is that the elements of $M_{x,0}^{D=0}$ are the formal horizontal sections of M centered at x. In the language of [25] and preceding literature on p-adic differential equations, one would think of x as the generic point of a certain subdisc of $\mathbb{D}_{\beta,K}$.

Following this intuition, one observes that for $y = H(x,\sigma)$ for some $\sigma > \rho(x)$, the discs of radius ρ centered at x and y coincide for all $\rho \in (\sigma,\beta)$. Formally, for any field L containing both $\mathcal{H}(x)$ and $\mathcal{H}(y)$, we obtain a natural isomorphism $L\langle (t-t_x)/\rho \rangle \cong L\langle (t-t_y)/\rho \rangle$. One consequence is that for $i \in \{1,\dots,n\}$, if $s_i(M,x) > \rho(x)$, then $s_i(M,x) = s_i(M, H(x,\rho))$ for all $\rho < s_i(M,x)$.

The relationship between radii of optimal convergence and intrinsic subsidiary radii (due in its original form to Young) is the following.

Definition 4.3.5. For $x \in \mathbb{D}_{\beta,K}$ not of type 1, let F_x be a copy of $\mathcal{H}(x)$ viewed as a differential field for the derivation $\frac{d}{dt}$.

Proposition 4.3.6. *For $x \in \mathbb{D}_{\beta,K}$ not of type 1, the intrinsic subsidiary radii of $M \otimes_{R_{[0,\beta]}} F_x$ are given by*

$$\min\{1, s_i(M,x)/\rho(x)\} \qquad (i = 1, \dots, n).$$

Proof. By Lemma 4.3.3, we may reduce to the case $x = \zeta_{0,\rho}$ for some $\rho \in (0,\beta]$. In this case, the claim follows from [25, Theorem 11.9.2]. □

One can also interpret Dwork's transfer theorem in this language.

Proposition 4.3.7. *For M nonzero, for all $x \in \mathbb{D}_{\beta,K}$ and $\rho \in [0,\beta]$,*

$$s_1(M, H(x,\rho)) \leq s_1(M,x).$$

Proof. Using Lemma 4.3.3, we may reduce to the case where $x = \zeta_{0,0}$, in which case the claim asserts that $s_1(M, \zeta_{0,\rho}) \leq s_1(M, \zeta_{0,0})$ for any $\rho \in [0, \beta]$. If $s_1(M, \zeta_{0,\rho}) > \rho$, then this follows from Remark 4.3.4. If $s_1(M, \zeta_{0,\rho}) \leq \rho$, then by Proposition 4.3.6, $\rho^{-1} s_1(M, \zeta_{0,\rho})$ equals the intrinsic radius of $M \otimes_{R_{[0,\beta]}} F_\rho$, so we may apply [25, Theorem 9.6.1] to conclude. □

Remark 4.3.8. The radius of convergence of M at any $x \in \mathbb{D}_{\beta,K}$ is always positive. This can be deduced either from Proposition 4.3.7 or from Clark's p-adic Fuchs theorem [25, Theorem 13.2.3]; the latter also covers the case of a regular singularity with p-adic non-Liouville exponent differences.

Remark 4.3.9. For M nonzero, the properties of the intrinsic radius described in Definition 2.2.2 carry over to the radius of convergence, as follows.

(a) We have $s_1(M^\vee, x) = s_1(M, x)$.
(b) For any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$, $s_1(M, x) = \min\{s_1(M_1, x), s_1(M_2, x)\}$.
(c) For any M_1, M_2, $s_1(M_1 \otimes M_2, x) \geq \min\{s_1(M_1, x), s_1(M_2, x)\}$, with equality if $s_1(M_1, x) \neq s_1(M_2, x)$.

However, unlike for intrinsic subsidiary radii, these properties do not propagate to radii of optimal convergence despite the validity of Proposition 4.3.6. The difficulty already appears in (a): the existence of a horizontal section of M on a large open disc does not imply the same for $M^\vee$. A similar difficulty arises for (b) unless we restrict consideration to split exact sequences. No such difficulty arises for (c).

So far we have consider only radii of convergence on closed discs, but one can make similar definitions for open discs.

Definition 4.3.10. For M a differential module over $R_{[0,\beta)}$, we may similarly define $s_i(M, x)$ for $x \in \mathbb{D}^\circ_{\beta,K}$. Then Proposition 4.3.7 implies that for M nonzero, for all $x \in \mathbb{D}^\circ_{\beta,K}$,

$$\limsup_{\rho \to \beta^-} s_1(M, H(x, \rho)) \leq s_1(M, x).$$

For open discs, we have the following key lemma.

Lemma 4.3.11. *Let M be a differential module over $R_{[0,\beta)}$. Suppose that for some $\gamma \in (0, \beta]$, there exists $m \in \{0, \dots, n\}$ satisfying the following conditions.*

(a) *For $i = 1, \dots, m$, $s_i(M, \zeta_{0,\rho})$ is constant and less than γ for ρ in some punctured left neighborhood of γ.*

(b) *For* $i = m+1, \dots, n$, $\limsup_{\rho \to \gamma^-} s_i(M, \zeta_{0,\rho}) \geq \gamma$.

Then the restrictions of the functions $s_i(M, \cdot)$ *to* $\mathbb{D}^\circ_{\gamma,K}$ *are constant for* $i = 1, \dots, n$.

Proof. Put $\alpha = \rho(x) > 0$. Using Proposition 4.3.6 to see that the appropriate hypotheses are satisfied, we may decompose $M \otimes_{R_{[0,\beta)}} R_{[0,\gamma)} = M_0 \oplus M_1 \oplus \cdots$ as per Corollary 3.6.8.

Consider any $k > 0$. By Corollary 3.6.8 and Proposition 4.3.6, for $i \in \{1, \dots, \operatorname{rank}(M_k)\}$, for all $y \in \mathbb{D}^\circ_{\gamma,K}$ we have $\min\{\rho(y), s_i(M_k, y)\} = \min\{\rho(y), e^{-c_k}\}$. For those y with $\rho(y) > e^{-c_k}$, we have

$$e^{-c_k} = \min\{\rho(y), e^{-c_k}\} = \min\{\rho(y), s_i(M_k, y)\}$$

and the right side cannot equal $\rho(y)$, so we must have $s_i(M_k, y) = e^{-c_k}$. For those y with $\rho(y) \leq e^{-c_k}$, we cannot have $s_i(M_k, y) > e^{-c_k}$: otherwise, we could choose $\delta \in (e^{-c_k}, s_i(M_k, y))$ and apply Remark 4.3.4 to see that $s_i(M_k, H(y, \delta)) = s_i(M_k, y) > e^{-c_k}$, contradicting the previously established equality $s_i(M_k, H(y, \delta)) = e^{-c_k}$. We thus have $s_i(M_k, y) \leq e^{-c_k}$; on the other hand, for any $\delta \in (e^{-c_k}, \alpha)$ we may apply Proposition 4.3.7 to obtain $e^{-c_k} = s_1(M_k, H(y, \delta)) \leq s_1(M_k, y) \leq s_i(M_k, y)$. We conclude that $s_i(M_k, y)$ is constant for $y \in \mathbb{D}^\circ_{\gamma,K}$.

For $i = 1, \dots, n$ and $y \in \mathbb{D}^\circ_{\gamma,K}$, we have

$$s_i(M \otimes_{R_{[0,\beta)}} R_{[0,\gamma)}, y) = \min\{s_i(M, y), \gamma\}. \tag{4.3.11.1}$$

For $i = 1, \dots, m$, we must have $s_i(M, y) < \gamma$ or else Remark 4.3.4 would lead to a violation of hypothesis (a); moreover, from (4.3.11.1) and the previous paragraph, $\min\{s_i(M, y), \gamma\}$ is constant on $\mathbb{D}^\circ_{\gamma,K}$. We are thus done in case $m = n$, so we may assume $m < n$ hereafter.

By Corollary 3.6.8, Proposition 4.3.6, and Proposition 4.3.7 (applied as in Definition 4.3.10), we have $s_1(M_0, y) \geq \gamma > \rho(y)$ for all $y \in \mathbb{D}^\circ_{\gamma,K}$. From this inequality plus (4.3.11.1), it follows that for $i = m+1, \dots, n$, we have $s_i(M, y) \geq \gamma$ for all $y \in \mathbb{D}^\circ_{\gamma,K}$. If there exists $y \in \mathbb{D}^\circ_{\gamma,K}$ for which $s_i(M, y) > \gamma$, then by Remark 4.3.4, $s_i(M, y)$ is constant on $\mathbb{D}^\circ_{\gamma,K}$; otherwise, $s_i(M, y)$ is evidently equal to the constant value γ on $\mathbb{D}^\circ_{\gamma,K}$. This completes the proof. $\square$

4.4. **Solvable modules.** If one views solvability of a differential module on an annulus as a question about what happens as one approaches the generic point of the inner boundary, one is then led to an analogous concept in which one approaches an arbitrary point of a Berkovich disc. For points of type 2, this amounts to a cosmetic revision of §3.8, but at points of other types one has more precise results. The case of type 4 points is especially critical in order to eliminate such points from the controlling graph of M (see Theorem 4.5.15).

Definition 4.4.1. Choose $x \in \mathbb{D}_{\beta,K}$ with $x \neq \zeta_{0,\beta}$, so that $\rho(x) < \beta$. Put $r_0 = -\log \rho(x)$. For any γ, δ with $\rho(x) < \gamma \leq \delta \leq \beta$, the subset of $\mathbb{D}_{\beta,K}$ consisting of points dominated by $H(x, \rho)$ for some $\rho \in [\gamma, \delta]$ has the form $\mathcal{M}(R_{x,[\gamma,\delta]})$ for some Banach algebra $R_{x,[\gamma,\delta]}$ over K. (More precisely, this subset is an *affinoid subdomain* of $\mathbb{D}_{\beta,K}$ in the sense of Definition 5.1.1.) For $\delta \in (\rho(x), \beta]$, define

$$R_{x,(\rho(x),\delta]} = \bigcap_{\gamma \in (\rho(x),\delta]} R_{x,[\gamma,\delta]}.$$

Define the *Robba ring at* x as the ring

$$\mathcal{R}_x = \bigcup_{\delta \in (\rho(x),\beta]} R_{x,(\rho(x),\delta]}.$$

All of these rings may be viewed as differential rings for the derivation $\frac{d}{dt}$.

Definition 4.4.2. For N a differential module of rank n over $\mathcal{R}_x$, the germ of the function $-\log s_i(N, H(x, e^{-r}))$ in a left neighborhood of r_0 is well-defined. We may thus say that N is *solvable at* x if

$$\limsup_{r \to r_0^-} -\log s_1(N, H(x, e^{-r})) - r \leq 0.$$

In this case, as in Definition 3.8.3, there exist nonnegative rational numbers $b_1(N, x) \geq \cdots \geq b_n(N, x)$ such that for $i = 1, \dots, n$, at the level of germs we have

$$\max\{r, -\log s_i(N, H(x, e^{-r}))\} = r + b_i(N, x)(r_0 - r).$$

Remark 4.4.3. For x of type 2, after making a finite extension of K to force K to be integrally closed in $\mathcal{H}(x)$, we may obtain an isomorphism $\mathcal{R}_x \cong \mathcal{R}_\alpha$ for $\alpha = \rho(x)$ by translating x to $\zeta_{0,\alpha}$. We may thus transfer statements about $\mathcal{R}_\alpha$, such as Theorem 3.8.17, directly to the setting of solvable modules over $\mathcal{R}_x$.

For x of other types, the behavior of a solvable module over $\mathcal{R}_x$ is much more restricted, especially in the case of a module obtained by base extension from $R_{[0,\beta]}$. It is most convenient to postpone discussion of this point until after we have Theorem 4.5.15 in hand; see §4.6. However, one key case is needed for the proof of Theorem 4.5.15, so we include it here; see Lemma 4.4.5.

Lemma 4.4.4. *Assume $p > 0$. Let $x \in \mathbb{D}_{\beta,K}$ be a point of type 4 for which $\rho(x) \in |\mathbb{C}^\times|$. Let N be a differential module over $\mathcal{R}_x$ of rank n which is solvable at x. Then $b_i(N, x) \in [0, 1]$ for $i = 1, \dots, n$.*

Proof. Using Lemma 4.3.3 and the fact that Berkovich's classification is preserved by passage from K to $\mathbb{C}$ (see Proposition 4.2.6), we may assume without loss of generality that $K = \mathbb{C}$ and $\rho(x) = 1$. We may also assume $n > 0$.

Let L_1, L_2 be two copies of $\mathcal{H}(x)$, and let L_3 be a complete extension of both (obtained by choosing an element of $\mathcal{M}(L_1 \widehat{\otimes}_K L_2)$). Let t_1, t_2 be the copies of t_x in L_1, L_2. For $i = 1, 2$, let $\mathcal{R}_{(i)}$ be a copy of $\mathcal{R}_1$ (that is, the ring $\mathcal{R}_\alpha$ with $\alpha = 1$) over L_i in the variable $t - t_i$. Let $\mathcal{R}_{(3)}$ be a copy of $\mathcal{R}_1$ over L_3 in the variable $t - t_1$, and equip $\mathcal{R}_{(3)}$ with the map from $\mathcal{R}_{(1)}$ sending $t - t_1$ to $t - t_1$ and the map from $\mathcal{R}_{(2)}$ sending $t - t_2$ to $t - t_1 + (t_1 - t_2)$. For $i = 1, 2$, we may identify the residue field of $\mathcal{R}_{(i)}$ with $\kappa_K((u_i))$ for $u_i = (t - t_i)^{-1}$, and then apply Theorem 3.8.17 and Corollary 3.8.18 to produce the minimal finite étale extension S_i of $\mathcal{R}_{(i)}^{\mathrm{int}}$ over which $N \otimes_{\mathcal{R}_{(i)}^{\mathrm{int}}} S_i$ satisfies the Robba condition. By the uniqueness in Corollary 3.8.18, we must then have an isomorphism

$$S_1 \otimes_{\mathcal{R}_{(1)}^{\mathrm{int}}} \mathcal{R}_3^{\mathrm{int}} \cong S_2 \otimes_{\mathcal{R}_{(2)}^{\mathrm{int}}} \mathcal{R}_3^{\mathrm{int}} \tag{4.4.4.1}$$

which commutes with the cocycle condition. This implies (e.g., by faithfully flat descent) that S_1 admits an action of the group of κ_K-linear substitutions on $\kappa_K((u_1))$ of the form $t \mapsto t + c$ with $c \in \kappa_K$. Let L be the residue field of S_1; applying Proposition 1.2.6, we may deduce that the highest upper numbering ramification break of L as a finite extension of $\kappa_K((u_1))$ is at most 1. By Corollary 3.8.18, this implies that $b_1(N, x) \leq 1$ and hence $b_i(N, x) \leq 1$ for $i = 1, \dots, n$. $\square$

Lemma 4.4.5. *Assume $p > 0$. Let M be a differential module over $R_{[0,\beta]}$ of rank n. Let x be a point of type 4 for which $\rho(x) \in |\mathbb{C}^\times|$. Put $N = M \otimes_{R_{[0,\beta]}} \mathcal{R}_x$. If N is solvable at x, then $b_i(N, x) \in \{0, 1\}$ for $i = 1, \dots, n$.*

Proof. Let $j \in \{0, \dots, n\}$ be any index for which $b_1(N, x), \dots, b_j(N, x) > 0$. For r in some left neighborhood of $-\log \rho(x)$, the function

$$\sum_{i=1}^{j} -\log s_i(M, H(x, e^{-r}))$$

is affine with nonpositive slope by Proposition 3.6.3(a,d). However, this slope is equal to

$$\sum_{i=1}^{j} (1 - b_i(N, x)),$$

each summand of which is nonnegative by Lemma 4.4.4. This proves the claim. $\square$

Remark 4.4.6. It is tempting to argue directly that the isomorphism (4.4.4.1) from the proof of Lemma 4.4.5 implies by faithfully flat descent that S_1 descends to a finite étale algebra over $\mathcal{R}_x^{\mathrm{int}}$. One obstruction to this approach is that it is unclear whether the maps $\mathcal{R}_x^{\mathrm{int}} \to \mathcal{R}_{(i)}^{\mathrm{int}}$ are flat.

4.5. **Controlling graphs for radii of convergence.** Using Proposition 4.3.6, we can give a partial translation of Proposition 3.6.3 into the language of radii of optimal convergence. This reproduces and improves a result of Pulita [36, Theorem 4.7] and of Baldassarri and the author [7]; see Remark 4.5.16. Throughout §4.5, retain Hypothesis 4.3.1.

Definition 4.5.1. For $x \in \mathbb{D}_{\beta,K}$, put

$$f_i(M,x) = -\log s_i(M,x) \qquad (i \in \{1,\dots,n\})$$

and $F_i(M,x) = f_1(M,x)+\cdots+f_i(M,x)$. Note that unlike the functions $f_i(M,r)$ considered in §3.6, the function $f_i(M,x)$ may take values less than $-\log \rho(x)$. We are thus led to define the truncated functions

$$\overline{s}_i(M,x) = \min\{\rho(x), s_i(M,x)\}$$
$$\overline{f}_i(M,x) = -\log \overline{s}_i(M,x).$$

Remark 4.5.2. By Proposition 4.3.6, the function $f_i(M,r)$ of §3.6 coincides with $\overline{f}_i(M, H(\zeta_{0,0}, e^{-r}))$. This will allow us to apply Proposition 3.6.3 to obtain information about the functions f_i.

Proposition 4.5.3. *For any $x \in \mathbb{D}_{\beta,K}$, for $i = 1,\dots,n$, $s_i(M,x)$ belongs to $|\mathbb{C}^\times|$ (which is the divisible closure of $|K^\times|$).*

Proof. By Remark 4.3.8, $s_i(M,x) > 0$. By Remark 4.3.4, for any $\alpha \in (0, s_i(M,x))$ and any y dominated by $H(x,\alpha)$, we have $s_i(M,y) = s_i(M,x)$. We may thus reduce to the case where x is of type 1. By Lemma 4.3.3, we may reduce to the case where $K = \mathbb{C}$ and $x = \zeta_{0,0}$.

By Remark 4.5.2 and Proposition 3.6.6, the function $\overline{f}_i(M, H(x, e^{-r}))$ is piecewise of the form $ar + b$ with $a \in \mathbb{Q}$ and $b \in \log |K^\times|$. Put $r_0 = -\log s_i(M,x)$. By Remark 4.3.4, r_0 is the largest value for which $\overline{f}_i(M, H(x, e^{-r})) = r$ for all $r \geq r_0$. We thus have $r_0 = -b/a$ for some $a \in \mathbb{Q}$ and $b \in \log |K^\times|$, proving the claim. □

Lemma 4.5.4. *For $i = 1,\dots,n$, if $s_i(M,x) > \rho(x)$ for some x, then $s_i(M,x)$ is constant on some neighborhood of x.*

Proof. This is immediate from Remark 4.3.4. □

Lemma 4.5.5. *For $i = 1,\dots,n$, the restriction of $f_i(M,\cdot)$ to any skeleton of $\mathbb{D}_{\beta,K}$ is piecewise affine.*

Proof. It suffices to check that for any $x \in \mathbb{D}_{\beta,K}$ not of type 1, the function g_i given by $g_i(r) = f_i(M, H(x, e^{-r}))$ is piecewise affine (the same then holds for points of type 1 by Lemma 4.5.4 and Remark 4.3.8). We first verify that $\max\{r, g_i(r)\} = \overline{f}_i(M, H(x, e^{-r}))$ is piecewise affine. Using Lemma 4.3.3, we may reduce to the case where $x = \zeta_{0,\alpha}$ for some $\alpha > 0$, in which case the claim follows from Proposition 3.6.3(a).

Given that $\max\{r, g_i(r)\}$ is piecewise affine, it follows that g_i is piecewise affine at any r_0 for which $g_i(r_0) > r_0$. At a value r_0 where $g_i(r_0) < r_0$, by Lemma 4.5.4, g_i is constant in a neighborhood of r_0. It thus suffices to check piecewise affinity at an arbitrary value r_0 at which $g_i(r_0) = r_0$.

We first consider a right neighborhood of r_0. If the values of r in this neighborhood for which $g_i(r) < r$ fail to accumulate at r_0, then in some smaller neighborhood we have $g_i(r) = r$ identically. Otherwise, for each value r_1 at which $g_1(r_1) < r_1$, by the previous paragraph g_i is constant for $r \geq r_1$. It follows that g_i is constant for $r > r_0$ and the constant value must be at most r_0. If it were strictly less than r_0, we would have $g_i(r_0) < r_0$ by Remark 4.3.4, contrary to hypothesis; we thus have $g_i(r) = r_0$ for $r \geq r_0$. This proves affinity to the right of r_0.

We next consider a left neighborhood of r_0. If there exists any r_1 in this neighborhood for which $g_i(r_1) < r_1$, then as above, g_i would be constant for $r \geq r_1$. But then we would have $r_0 = g_i(r_0) = g_i(r_1) < r_1 < r_0$, a contradiction. Hence $g_i(r) = r$ identically in this neighborhood. This proves affinity to the left of r_0. □

Remark 4.5.6. By Lemma 4.5.5, it makes sense to refer to the slopes of $f_i(M, \cdot)$ or $\overline{f}_i(M, \cdot)$ along any branch of $\mathbb{D}_{\beta,K}$.

Definition 4.5.7. For $x \in \mathbb{D}_{\beta,K}$, define the *spectral cutoff* of M at x to be the largest value $m(x) \in \{0, \dots, n\}$ such that $s_i(M, x) < \rho(x)$ for $i = 1, \dots, m(x)$.

Lemma 4.5.8. *Let U be a lower branch of $\mathbb{D}_{\beta,K}$ at some point x. Suppose that for $i = 1, \dots, m(x)$, the slope of $\overline{f}_i(M, \cdot)$ along U (which exists by Lemma 4.5.5) is equal to* 0. *Then*

$$s_i(M, x) = s_i(M, y) \qquad (y \in U;\ i = 1, \dots, n).$$

Proof. By Lemma 4.5.5, for any $y \in U$ we have $s_i(M, H(y, \rho)) \to s_i(M, x)$ as $\rho \to \rho(x)^-$. This implies on one hand that the conditions of Lemma 4.3.11 are satisfied, so $s_i(M, y)$ is constant for $y \in U$, and on the other hand that this constant value is equal to $s_i(M, x)$. □

Lemma 4.5.9. *Let S be a rooted skeleton of $\mathbb{D}_{\beta,K}$. Let T be the interior of an edge in a subdivision of S. Suppose that for $i = 1, \dots, n$, $\overline{f}_i(M, \cdot)$*

is affine on T. Then

$$s_i(M, y) = s_i(M, \pi_S(y)) \qquad (i = 1, \dots, n; y \in \pi_S^{-1}(T)).$$

Proof. For $x \in T$ and $i = 1, \dots, m(x)$, by Proposition 3.6.3(c,d) and Remark 4.5.2, the slope of $\overline{f}_i(M, \cdot)$ along any lower branch of x other than the one meeting T is equal to 0. The claim thus follows from Lemma 4.5.8. □

Lemma 4.5.10. *For any $x \in \mathbb{D}_{\beta,K}$, along all but finitely many lower branches of $\mathbb{D}_{\beta,K}$ at x, the slope of $\overline{f}_i(M, \cdot)$ is 0 for $i = 1, \dots, m(x)$.*

Proof. This is immediate from Proposition 3.6.3(c). □

Lemma 4.5.11. *For any $x \in \mathbb{D}_{\beta,K}$, there exist a skeleton S of $\mathbb{D}_{\beta,K}$ and an open neighborhood I of $\pi_S(x)$ such that the restrictions of $s_1(M, \cdot), \dots, s_n(M, \cdot)$ to $\pi_S^{-1}(I)$ factor through π_S. Moreover, we may choose S to have no generators of type 3.*

Proof. By Lemma 4.5.10, along all but finitely many lower branches of $\mathbb{D}_{\beta,K}$ at x, the slope of $\overline{f}_i(M, \cdot)$ is 0 for $i = 1, \dots, m(x)$. Choose S to pass through x and meet each of the remaining lower branches of X at x; this can always be done without using generators of type 3 because any point of type 3 dominates some points of type 2 by Proposition 4.2.6). By Lemma 4.5.5, we can find a subdivision of S such that for $i = 1, \dots, n$, $\overline{f}_i(M, \cdot)$ is affine on each edge of the subdivision meeting x. Let I be the union of the interiors of these edges, together with x. For $y \in \pi_S^{-1}(I)$, we have $s_i(M, y) = s_i(M, \pi_S(y))$ by Lemma 4.5.8 (if $\pi_S(y) = x$) or Lemma 4.5.9 (if $\pi_S(y) \neq x$). □

Lemma 4.5.12. *For $x \in \mathbb{D}_{\beta,K}$ of type 4, for $i = 1, \dots, n$, in some left neighborhood of $\rho(x)$, the function*

$$\rho \mapsto \min\{\omega\rho, s_i(M, H(x, \rho))\}$$

is either constant or identically equal to $\omega\rho$.

Proof. Apply Corollary 2.1.6 to construct $\mathbf{v} \in M$ which is a cyclic vector in $M \otimes_{R_{[0,\beta]}} \mathrm{Frac}(R_{[0,\beta]})$, and write $D^n(\mathbf{v}) = a_0\mathbf{v} + \cdots + a_{n-1}D^{n-1}(\mathbf{v})$ for some $a_0, \dots, a_{n-1} \in \mathrm{Frac}(R_{[0,\beta]})$. Since x is of type 4, for $i = 0, \dots, n-1$, the function $y \mapsto y(a_i)$ is constant in some neighborhood of x. By Proposition 2.2.6, this yields the desired result. □

Lemma 4.5.13. *For $x \in \mathbb{D}_{\beta,K}$ of type 4, for $i = 1, \dots, n$, in some left neighborhood of $\rho(x)$, the function $\rho \mapsto \overline{s}_i(M, H(x, \rho))$ is either constant or identically equal to ρ.*

Proof. This is immediate from Lemma 4.5.13 if $p = 0$, so we may assume $p > 0$; we may also assume $K = \mathbb{C}$. Let h be the smallest nonnegative integer for which $s_i(M, x) \notin (\omega^{p^{-h-1}}\rho(x), \rho(x))$ for $i = 1, \dots, n$. We proceed by induction on h.

Put $r_0 = -\log \rho(x)$; since x is of type 4, we have $r_0 > -\log \beta$. Let $j \in \{0, \dots, n\}$ be the largest value for which $s_i(M, x) \leq \omega\rho(x)$ for $i = 1, \dots, j$. Since the functions $r \mapsto \overline{f}_i(M, H(x, e^{-r}))$ are continuous by Lemma 4.5.5, we may apply Lemma 4.5.13 to produce $r_1 \in (-\log \beta, r_0)$ such that for $i = 1, \dots, j$, the function $r \mapsto f_i(M, H(x, e^{-r}))$ is constant for $r \in [r_1, r_0]$. By moving r_1 towards r_0, we may also ensure that $\rho(x) > \omega e^{-r_1}$ and $s_i(M, H(x, e^{-r_1})) > \omega e^{-r_1}$ for $i > j$. By rescaling t, we may further ensure that $r_1 < 0 < r_0$.

By Proposition 4.2.6, we can find $z \in \mathbb{C}$ such that $H(x, 1) = \zeta_{z,1}$. There is no harm in applying a translation on the disc to reduce to the case $z = 0$. If we put $\beta' = e^{-r_1}$, then by Proposition 3.6.7, the restriction of M to $\mathbb{D}^\circ_{\beta',K}$ splits as a direct sum $M_1 \oplus M_2$ with $\mathrm{rank}(M_1) = j$ and $f_i(M, e^{-r}) = f_i(M_1, e^{-r})$ for $i = 1, \dots, j$ and $r \in (r_1, 0]$. By Corollary 3.6.5, the original claim holds with M replaced by the restriction of M_1 to $\mathbb{D}_{1,K}$.

Let N be the restriction of M_2 to $\mathbb{D}_{1,K}$; it now suffices to prove the original claim with M replaced by N. We may assume $j < n$, as otherwise there is nothing to check. We first check the claim for N in case $s_{i+1}(M, x) \geq \rho(x)$, which in particular will cover the base case $h = 0$ of the induction. If $\rho(x) \notin |\mathbb{C}^\times|$, then Proposition 4.5.3 implies that $s_i(N, x) > \rho(x)$ for all i, so the desired result follows by Lemma 4.5.4. If instead $\rho(x) \in |\mathbb{C}^\times|$, then the desired result follows by Lemma 4.4.5.

We next check the claim for N in case $s_{i+1}(M, x) < \rho(x)$; note that by construction we also have $\omega\rho(x) < s_{i+1}(M, x)$. Let $\psi : \mathbb{D}^\circ_{1,K} \to \mathbb{D}^\circ_{1,K}$ be the map for which $\psi^*(t) = (t + 1)^p - 1$. Put $y = \psi(x)$; it is a point of type 4 with $\rho(y) = \rho(x)^p$. Let N' be the off-center Frobenius descendant of N in the sense of Proposition 3.5.5 with $\lambda = 1$. By that proposition, $\overline{s}_{(p-1)(n-j)+i}(N', z) = \overline{s}_i(N, z)^p$ for $i = 1, \dots, n-j$ and $z \in \mathbb{D}_{1,K}$ with $\rho(z) > \omega$. Since we assumed that $\rho(x) > \omega\beta' > \omega$, we have $\overline{s}_{(p-1)(n-j)+i}(N', H(y, \rho^p)) = \overline{s}_i(N, H(x, \rho))^p$ for $i = 1, \dots, n-j$ and $\rho \in [\rho(x), 1]$. We may thus deduce the claim for N from the corresponding claim for N', to which we may apply the induction hypothesis because we have decreased the value of h. □

Lemma 4.5.14. *For $x \in \mathbb{D}_{\beta,K}$ of type 1 or 4, for $i = 1, \dots, n$, the function $s_i(M, \cdot)$ is constant on some neighborhood of x.*

Proof. For x of type 1, the claim follows from Remark 4.3.4 and Remark 4.3.8. For x of type 4, Lemma 4.5.13 implies that the hypothesis of Lemma 4.5.8 holds for some open disc containing x, yielding the claim in this case. $\square$

Theorem 4.5.15. (a) *There exists a strict skeleton S of $\mathbb{D}_{\beta,K}$ such that $s_1(M,\cdot),\dots,s_n(M,\cdot)$ factor through π_S.*
(b) *For $i = 1,\dots,n$, $f_i(M,\cdot)$ is piecewise affine with slopes in $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$. Moreover, $F_n(M,\cdot)$ has integral slopes.*
(c) *There is a unique minimal graph G in $\mathbb{D}_{\beta,K}$ which is a controlling graph for all of the functions $f_i(M,\cdot)$. Moreover, the vertices of G are all of type 2. (We call G the* controlling graph *of M.)*

Proof. For each $x \in \mathbb{D}_{\beta,K}$, apply Lemma 4.5.11 to construct a skeleton S_x of $\mathbb{D}_{\beta,K}$ and and an open neighborhood I_x of $\pi_S(x)$ such that the restrictions of $s_1(M,\cdot),\dots,s_n(M,\cdot)$ to $\pi_{S_x}^{-1}(I_x)$ factor through π_{S_x}. Since $\pi_{S_x}^{-1}(I_x)$ is open in the compact space $\mathbb{D}_{\beta,K}$, we can choose finitely many points $x_i \in \mathbb{D}_{\beta,K}$ such that, if we relabel S_x, I_x as S_i, I_i, then the open sets $\pi_{S_i}^{-1}(I_i)$ cover $\mathbb{D}_{\beta,K}$. Let S be the union of the S_i; for $y \in \pi_{S_i}^{-1}(I_i)$, we have $\pi_{S_i}(y) = \pi_{S_i}(\pi_S(y))$ and so $s_i(M,y) = s_i(M,\pi_{S_i}(y)) = s_i(M,\pi_S(y))$. This proves (a) except that S might include some generators of types 1 or 4 (generators of type 3 are excluded by Lemma 4.5.11). However, by Lemma 4.5.14, if x is a generator of type 1 or 4, then the functions $s_i(M,\cdot)$ are constant in a neighborhood of x, so we may replace x with a point of type 2 in this neighborhood which dominates x. We thus deduce (a).

From (a), we deduce piecewise affinity using Lemma 4.5.5. To deduce integrality of slopes, we apply Proposition 3.6.3(b) at points x where $s_i(M,x) < \rho(x)$ and Lemma 4.5.4 at points x where $s_i(M,x) > \rho(x)$. This fails to account for segments where $s_i(M,x) = \rho(x)$ identically, but on any such segment $f_i(M,x)$ has slope 1. We thus deduce (b).

Using (a) and (b), we deduce the existence of the minimal controlling graph G and the fact that none of its vertices is of type 1 or 4. Using Proposition 4.5.3, we further deduce that G has no vertices of type 3, thus yielding (c). $\square$

Remark 4.5.16. The weaker form of Theorem 4.5.15 in which strictness of the skeleton is not asserted is the main result of [7] applied to a disc, with materially the same proof as given above. It is also the essential content of [36, Theorem 4.7] applied to a disc: more precisely, parts (i) (finiteness) and (ii) (integrality) of that result are included in Theorem 4.5.15. The proof in [36] is a bit different, making use

of a combinatorial criterion for piecewise affinity. Neither the analysis in [7] nor in [36] includes any special study of type 4 points, as these are treated by base extension to convert them into other types. Consequently, the techniques of those paper cannot by themselves exclude vertices of type 4 from the controlling graph, which here is made possible by the analysis in §4.4.

Note that [36, Theorem 4.7] gives a finer description of the controlling graph than appears either here or in [7]. It also includes weak analogues of the convexity, subharmonicity, and monotonicity assertions from Proposition 3.6.3 (although with a change of sign convention, so convexity becomes concavity and subharmonicity becomes superharmonicity). In [36, Theorem 4.7] these statements are used in an essential way to prove finiteness; however, given Theorem 4.5.15, they can be deduced directly from Proposition 3.6.3.

Note also that [7] and [35, 36] consider not just discs but more general curves. We will return to this more general case in §5.

4.6. **More on solvable modules.** With Theorem 4.5.15 in hand, we can now fill out the discussion of solvable modules over $\mathcal{R}_x$ initiated in §4.4. We also point out a link with our previous work on semistable reduction for overconvergent F-isocrystals [28].

Hypothesis 4.6.1. Throughout §4.6, let M be a differential module over $R_{[0,\beta]}$ of rank n, choose $x \in \mathbb{D}_{\beta,K}$, put $M_x = M \otimes_{R_{[0,\beta]}} \mathcal{R}_x$, and let N be a subquotient of M_x which is solvable at x.

Remark 4.6.2. For x of type 3, Theorem 4.5.15 forces N to satisfy the Robba condition; if $N = M_x$, then N is forced to be trivial by Proposition 4.3.7. For x of type 1, we can say even more: Proposition 4.3.7 and Theorem 4.5.15 together imply that M_x itself is a trivial differential module, as then is N.

For x of type 4, we have the following refinements of Lemma 4.4.5.

Proposition 4.6.3. *Suppose that x is of type 4.*

(a) *If $\rho(x) \in |\mathbb{C}^\times|$, then $b_i(N,x) \in \{0,1\}$ for all i.*
(b) *If $\rho(x) \notin |\mathbb{C}^\times|$, then N is trivial, so $b_i(N,x) = 0$ for all i.*

Proof. By Theorem 4.5.15 (or Lemma 4.5.14), the functions $s_i(M,\cdot)$ are constant in a neighborhood of x. This immediately implies (a). To deduce (b), note that we must have $s_i(M,x) \neq \rho(x)$ by Proposition 4.5.3. Since the $s_i(M,\cdot)$ are constant, we may apply Proposition 3.6.7 to decompose M in a neighborhood of x as a direct sum $M' \oplus M''$ with $s_i(M',x) < \rho(x)$ for all i and $s_i(M'',x) > \rho(x)$ for all i. In particular, M'' is trivial on some neighborhood of x; moreover, the projection of

N onto $M' \otimes \mathcal{R}_x$ must be zero. It follows that N is trivial, yielding (b). $\square$

Theorem 4.6.4. *Assume that $K = \mathbb{C}$, x is of type 4, and $\rho(x) = 1$. For each $c \in \kappa_K$, choose a lift $\tilde{c}$ of c to $\mathfrak{o}_K$, and let Q_c be the differential module over $\mathcal{R}_x$ free on one generator $\mathbf{v}$ such that $D(\mathbf{v}) = \tilde{c}\mathbf{v}$.*

(a) *For each irreducible subquotient P of N, there exists $c \in \kappa_K$ such that $P \otimes Q_c$ satisfies the Robba condition.*

(b) *There exists a finite étale extension S of $\mathcal{R}_x$ of the form*
$$S = \mathcal{R}_x[z_1, \dots, z_m]/(z_1^p - z_1 - a_1 t, \dots, z_m^p - z_m - a_m t)$$
for some nonnegative integer m and some $a_1, \dots, a_m \in \mathfrak{o}_K^\times$ such that $N \otimes_{\mathcal{R}_x} S$ is trivial.

Proof. By Proposition 4.6.3 we have $b_1(P) \in \{0, 1\}$. If $b_1(P) = 0$ we take $c = 0$; otherwise, by [25, Theorem 12.7.2], we can choose c so that $b_1(P \otimes Q_c) < 1$, and then by Proposition 4.6.3 again we have $b_1(P \otimes Q_c) = 0$. This proves (a).

Given (a), to prove (b), Theorem 4.5.15 and Proposition 3.6.7 allow us to reduce to the case where M_x itself is solvable at x; we may then further reduce to the case where $N = M_x$. In this case, the proof of Theorem 3.8.17 provides S such that $N \otimes_{\mathcal{R}_x} S$ satisfies the Robba condition. However, by induction on m, we see that there is an isomorphism
$$R_{[0,\beta^{p^{-m}}]} \cong R_{[0,\beta]}[z_1, \dots, z_m]/(z_1^p - z_1 - a_1 t, \dots, z_m^p - z_m - a_m t) \tag{4.6.4.1}$$
sending t to z_m. This isomorphism gives rise to a map $\psi : \mathbb{D}_{\beta^{p^{-m}},K} \to \mathbb{D}_{\beta,K}$ by mapping $R_{[0,\beta]}$ into the right side of (4.6.4.1) and then crossing to the left side. The inverse image of x under this map is a single point y. By construction, $N \otimes_{\mathcal{R}_x} S \cong \psi^* M \otimes_{R_{[0,\beta^{p^{-m}}]}} \mathcal{R}_y$ satisfies the Robba condition. By Proposition 4.3.7, $\psi^* M$ is trivial in a neighborhood of y, so $N \otimes_{\mathcal{R}_x} S$ is also trivial. $\square$

Corollary 4.6.5. *Assume that x is of type 4. Then any subquotient of N satisfying the Robba condition is trivial, and hence admits the zero tuple as an exponent.*

Proof. If $\rho(x) \notin |\mathbb{C}^\times|$, then N is trivial by Proposition 4.6.3(b), so any subquotient of N satisfying the Robba condition is also trivial and hence admits the zero tuple as an exponent. If $\rho(x) \in |\mathbb{C}^\times|$, we may assume that $K = \mathbb{C}$ and $\rho(x) = 1$. Set notation as in the proof of Theorem 4.6.4(b), again reducing to the case where $N = M_x$. In this case, the Tannakian category of differential modules over $\mathcal{R}_x$ generated by N admits a fibre functor computing horizontal sections over S, for

which the automorphism group is an elementary abelian p-group. In particular, N splits as a direct sum of irreducible submodules whose p-th tensor powers are trivial. Consequently, to check that a subquotient of N satisfying the Robba condition is trivial, it suffices to check the case of a irreducible submodule P for which $P^{\otimes p}$ is trivial; this case follows from Corollary 3.4.25. □

Remark 4.6.6. Note that the isomorphism in (4.6.4.1) depends critically on having linear powers of t on the right side; otherwise, we would end up with something other than a disc, so Dwork's transfer theorem (Proposition 4.3.7) would not apply. This is why it is necessary to invest the hard work to first prove $b_i(N, x) \in \{0, 1\}$ in order to deduce Corollary 4.6.5.

Remark 4.6.7. The above arguments, including the proof of Lemma 4.4.5, are loosely inspired by the arguments made in [28, §5]. However, the correspondence turns out to be somewhat less close than we had originally expected, primarily because the process of transposing the arguments exposed an error in [28]. We now describe this error and how it may be remedied using results from this paper.

The error appears in the second sentence of the proof of [28, Lemma 5.6.2]: it is not the case that the property of being terminally presented is stable under tame alterations. That is because the tame alteration $x \mapsto x^m$ is ramified along the segment joining 0 to the Gauss point; consequently, after pulling back a terminally presented module along a tame alteration, one encounters a change of slope at the point where one branches off from the ramification locus. In the continuation of the proof, the tame alteration is erroneously used to force the group $\tau(I_1)$, which initially is the semidirect product of the p-group $\tau(W_1')$ with a cyclic group of order prime to p, to become equal to $\tau(W_1')$.

To correct the proof, it suffices to establish that the equality $\tau(I_1) = \tau(W_1')$ holds initially, so that no tame alteration is needed and the rest of the argument may proceed unchanged. To verify this, choose ρ as in [25, Lemma 4.7.4]; by that lemma, $|\cdot|_{\rho^\alpha, s_0}$ defines a point of $\mathcal{M}(\ell\langle x\rangle)$ of type 4. We may thus apply Corollary 4.6.5 to deduce that any subquotient of the cross-section M_ρ which satisfies the Robba condition admits the zero tuple as an exponent. This implies that $\tau(W_1')$ has no nontrivial quotient of prime-to-p order, and so $\tau(I_1) = \tau(W_1')$ as desired.

One might prefer to incorporate some of the intermediate arguments from this paper into the proof method of [28], but this seems difficult. The plan of attack in [28] is to pick out an Artin-Schreier extension that reduces the image of the monodromy representation, which requires

tame ramification to be ruled out first. By contrast, the method here is to use Artin-Schreier extensions only to lower the ramification numbers; only when this stops being possible is the presence of tame ramification ruled out.

A more satisfying resolution would be to use additional results of this paper, especially Theorem 4.6.4, to shortcut many of the complicated proofs in [28, §5]. We leave this as an exercise for the interested reader.

5. Berkovich curves

To conclude, we globalize our setup to include more general Berkovich curves. We now adopt the full language of Berkovich analytic spaces, as in [8, 9].

5.1. **Analytic spaces.**

Definition 5.1.1. A *strictly affinoid algebra* (resp. an *affinoid algebra*) over K is a commutative Banach algebra over K isomorphic to a quotient of the completion of some polynomial ring $K[T_1, \dots, T_n]$ for the Gauss norm (resp. the $(r_1, \dots, r_n)$-Gauss norm for some $r_1, \dots, r_n > 0$).

Let A be a (strictly) affinoid algebra over K. A *(strictly) affinoid subdomain* of $\mathcal{M}(A)$ is a closed subset U for which the category of bounded K-linear homomorphisms $A \to B$ of (strictly) affinoid algebras whose restriction maps carry $\mathcal{M}(B)$ into U has an initial element. Any such initial homomorphism $A \to B$ is then flat and induces a homeomorphism $\mathcal{M}(B) \cong U$ [8, Proposition 2.2.4]; in particular, a strictly affinoid subdomain is also an affinoid subdomain.

Note that $\mathcal{M}(A)$ admits a neighborhood basis of affinoid subdomains, e.g., because any rational subdomain is an affinoid subdomain. For $x \in \mathcal{M}(A)$, define the local A-algebra A_x as the direct limit of the representing homomorphisms $A \to B$ over all affinoid subdomains of $\mathcal{M}(A)$ which are neighborhoods of x. We define the *structure sheaf* $\mathcal{O}$ on $\mathcal{M}(A)$ so that for U an open subset of $\mathcal{M}(A)$, $\mathcal{O}(U)$ consists of the functions $f : U \mapsto \coprod_{x \in \mathcal{M}(A)} A_x$ such that for each $x \in U$, there exist a homomorphism $A \to B$ and an element $g \in B$ such that:

- the map $A \to B$ represents an affinoid subdomain of $\mathcal{M}(A)$ contained in U and containing a neighborhood of x;
- for each $y \in U$, $f(y)$ is the image of g in A_y.

By Tate's theorem, the natural map $A \to \Gamma(\mathcal{M}(A), \mathcal{O})$ is a bijection. By Kiehl's theorem, coherent sheaves over $\mathcal{O}$ correspond to finite A-modules via the functor of global sections.

Definition 5.1.2. A *good (strictly) K-analytic space* is a locally ringed space which is locally isomorphic to an open subspace of the Gel'fand

spectrum of a (strictly) affinoid algebra over K. These are the analytic spaces considered in [8]; they have the property that any point has a neighborhood basis consisting of affinoid spaces.

Remark 5.1.3. In [9], the more general notion of a *(strictly) K-analytic space* is considered, in which it is only required that each point have a neighborhood basis consisting of a finite union of affinoid spaces (glued in a suitable way). In this paper, we can get away with considering only good spaces because any curve over K is good [16, Corollary 3.4].

5.2. **Curves and triangulations.** We next introduce some of the the combinatorial structure of a Berkovich analytic curve over K. One way to explain this is using semistable models, as in [5, 7]. Here, we take an alternate approach using triangulations introduced by Ducros [17], so as to avoid leaving the realm of analytic spaces; this follows the example of [36, 35]. There is also a link to tropicalization; see Remark 5.2.7.

Definition 5.2.1. For K' an analytic field containing K and X a good K-analytic space, let $X_{K'}$ denote the base extension of K to K'. For $X = \mathcal{M}(A)$, we have $X_{K'} = \mathcal{M}(A \widehat{\otimes}_K K')$.

Let Ω_X denote the sheaf of continuous Kähler differentials on X. We say that X is *rig-smooth of pure dimension n* if for every analytic field K' containing K, $\Omega_{X_{K'}}$ is locally free of rank n.

By a *curve* over K, we will mean a good K-analytic space X which is separated (i.e., the diagonal morphism is a closed immersion) and rig-smooth of pure dimension 1. In particular, X is paracompact.

Definition 5.2.2. Let X be a curve over K. For $x \in X$, we declare x to be of *type* $1, 2, 3, 4$ if the signature of x is respectively $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 0, 0)$. These cases are exhaustive by Proposition 4.2.6 plus Noether normalization for strictly affinoid algebras [10, Corollary 6.1.2/2].

Definition 5.2.3. An *open disc* over K is a K-analytic space isomorphic to $\bigcup_{\gamma \in (0,\beta]} \mathcal{M}(R_{[0,\gamma]})$ for some $\beta > 0$. An *open annulus* over K is a K-analytic space isomorphic to $\bigcup_{\alpha < \gamma \leq \delta < \beta} \mathcal{M}(R_{[\gamma,\delta]})$ for some $0 < \alpha < \beta$.

A *virtual open disc* (resp. *virtual open annulus*) is a connected K-analytic space whose base extension to $\mathbb{C}$ is a disjoint union of discs (resp. annuli). By the *skeleton* of a virtual open annulus over K, we mean the set of points not contained in a virtual open disc. For the standard open annulus $\bigcup_{\alpha < \gamma \leq \delta < \beta} \mathcal{M}(R_{[\gamma,\delta]})$ within $\mathbb{D}_{\beta,K}$, the skeleton

is the set $\{\zeta_{0,\rho} : \rho \in (\alpha, \beta)\}$; in general, the skeleton of a virtual open annulus is an open segment.

Definition 5.2.4. Let X be a curve over K. A *weak strict triangulation* (resp. *weak triangulation*) of X is a locally finite subset S of X consisting of points of type 2 (resp. of types 2 or 3) such that any connected component of $X \setminus S$ is a virtual open disc or a virtual open annulus. The union of the skeleta of the connected components of $X \setminus S$ which are virtual open annuli forms a locally finite graph Γ_S, called the *skeleton* of the weak triangulation. The points of Γ_S are all of types 2 or 3.

Remark 5.2.5. The definition of weak triangulation used here is the same as in [36] but is somewhat more permissive than the one used in [17], in which it is required that $X \setminus S$ be relatively compact. Omitting this condition makes it possible for Γ_S to fail to meet some connected components of X, e.g., if there is a component which is itself a virtual open disc. If Γ_S does meet every connected component of X, then there is a natural continuous retraction $\pi_S : X \to \Gamma_S$ taking any $x \in \Gamma_S$ to itself and taking any $x \in X \setminus \Gamma_S$ to the unique point of Γ_S in the closure of the connected component of $X \setminus \Gamma_S$ containing x.

Theorem 5.2.6. *Any (strictly) analytic curve over K admits a weak (strict) triangulation.*

Proof. See [17]. $\square$

Remark 5.2.7. There is also an approach to the structure theory of analytic curves via tropicalization, i.e., consideration of the projections defined by evaluation at finitely many functions on the curve. For discussion of the case $K = \mathbb{C}$, including a proof of Theorem 5.2.6 in that context, see [4, §5].

Definition 5.2.8. Let X be a curve and let $x \in X$ be a point of type 2. Then the residue field $\kappa_{\mathcal{H}(x)}$ is the function field of an algebraic curve over κ_K; we denote the genus of this function field by $g(x)$ and call it the *genus* of x. For any weak triangulation S of X, the type 2 points of $X \setminus S$ are all of genus 0; by Theorem 5.2.6, it follows that the type 2 points of X of positive genus form a locally finite set.

Definition 5.2.9. Let X be a curve. By a *branch* of X at a point $x \in X$, we mean a local path-connected component of $X \setminus \{x\}$ at x. Depending on the type of x, branches exist as follows.

1. Exactly one branch.
2. Infinitely many branches, corresponding to all but finitely many places of the function field $\kappa_{\mathcal{H}(x)}$.

3. Either one or two branches.
4. Exactly one branch.

Given a weak triangulation S of X, we say a branch U of X at x is *skeletal* (or *S-skeletal* in case of ambiguity) if the closure of $U \cap \Gamma_S$ contains x; such branches can only exist if $x \in \Gamma_S$.

We say that $x \in X$ is *external* if it is of type 2 and its branches do not correspond to all of the places of $\kappa_{\mathcal{H}(x)}$ or if it is type 3 and it has only one branch; otherwise, we say that x is *internal*. For any weak triangulation S of X, every point of $X \setminus \Gamma_S$ is internal, as is every point of Γ_S lying in the interior of an edge; by Theorem 5.2.6, it follows that the external points of X form a locally finite set.

Example 5.2.10. For X an affinoid space, the external points of X are precisely the points of the *Shilov boundary*, the minimal subset of X for which the maximal modulus principle holds.

5.3. **Convergence of local horizontal sections.** We next study the convergence of local horizontal sections on analytic curves. As in the case of discs, we end up with a global statement about the behavior of radii of convergence of differential modules on analytic curves; this statement recovers the main results of [7] and [36, 35].

Hypothesis 5.3.1. For the remainder of the paper, let X be a curve over K equipped with a weak triangulation S and let M be a vector bundle over X of constant rank $n > 0$ equipped with a connection. (Since X is of dimension 1, the connection is automatically integrable.)

Remark 5.3.2. One interesting case excluded by our hypotheses is that where X is an affine line and S is empty. In this case, the radii of convergence should be allowed to be infinite, but we do not want to worry about this.

In order to define analogues of the radii of optimal convergence, one must make reference to the chosen triangulation. This has the same effect as the choice of a semistable model in [5, 7].

Definition 5.3.3. For $x \in \Gamma_S$, define $s_1(M, S, x), \dots, s_n(M, S, x)$ as the intrinsic subsidiary radii of M in order, and put $\rho_S(x) = 1$.

For $x \in X \setminus \Gamma_S$, lift x to a point $y \in X_{\mathbb{C}}$, identify the connected component of $(X \setminus \Gamma_S)_{\mathbb{C}}$ containing x with an open unit disc, then define $s_1(M, S, x), \dots, s_n(M, S, x)$ as the functions $s_1(M, y), \dots, s_n(M, y)$ as in Definition 4.3.10. We use the same identification to define the diameter $\rho_S(x)$. These definitions do not depend on the choice of y, and are stable under enlarging K.

For $x \in X$, define the *spectral cutoff* of M as the largest $m(x) \in \{0, \dots, n\}$ such that $s_i(M, S, x) < \rho_S(x)$ for $i = 1, \dots, m(x)$.

In order to analyze these functions, it will be useful to consider them first along individual branches.

Definition 5.3.4. Choose $x \in \Gamma_S$ of type 2, let U be a branch of X at x, and let v be the corresponding place of $\kappa_{\mathcal{H}(x)}$. Choose $t \in \mathcal{O}_{X,x}$ with $x(t) = 1$ whose image $\overline{t}$ in $\kappa_{\mathcal{H}(x)}$ is a uniformizer of v (i.e., its v-valuation is the positive generator of the value group). Then for $\beta \in (0, 1)$ sufficiently close to 1, t defines an isomorphism between the space of $y \in U$ with $y(t) \in (\beta, 1)$ and the open annulus $\beta < |t| < 1$ in the t-line. We can use this isomorphism to define the class of functions $f : X \to \mathbb{R}$ which are affine along U in a neighborhood of x, and to associate to each such function a slope (in the direction away from x); neither of these definitions depends on the choice of t.

Lemma 5.3.5. *Set notation as in Definition 5.3.4. Then for $i = 1, \dots, m(x)$, the function $\log s_i(M, S, \cdot)$ is affine along U and its limit at x (approaching from within U) equals $\log s_i(M, S, x)$.*

Proof. For $x \notin S$ this is immediate from Proposition 3.6.3(a). For $x \in S$ with $g(x) = 0$, we may also apply Proposition 3.6.3(a) over the ring $R^{\mathrm{an}}_{(\alpha,1)}$. For $x \in S$ with $g(x) \neq 0$, we obtain a differential module over a ring S which can be written as a finite étale algebra over $R^{\mathrm{an}}_{(\alpha,1)}$ of some degree $d > 0$ such that $S \otimes_{R^{\mathrm{an}}_{(\alpha,1)}} F_1 \cong \mathcal{H}(x)$ is a finite *unramified* extension of F_1. If we restrict scalars from S to $R^{\mathrm{an}}_{(\alpha,1)}$, the multiset of intrinsic subsidiary radii does not change except that each multiplicity gets multiplied by d. We may thus apply Proposition 3.6.3(a) in this case also. □

We have the following analogue of Proposition 3.6.3(c).

Theorem 5.3.6. *Choose $x \in X$ of type 2. Let $c(x)$ be the number of skeletal branches of X at x. (Note that if $x \notin \Gamma_S$, then $g(x) = c(x) = 0$.)*

(a) *For $i = 1, \dots, m(x)$, the function $\log s_i(M, S, \cdot)$ is affine of slope 0 along all but finitely many branches of X at x. In particular, we may form the sum μ_i of the slopes of the function $\sum_{j=1}^{i} \log s_j(M, S, \cdot)$ along all of the branches of X at x (in the directions away from x).*
(b) *If $x \notin \Gamma_S$, then $\mu_i \leq 0$ for $i = 1, \dots, m(x)$.*
(c) *If $x \in \Gamma_S$ is internal, then $\mu_i \leq (2g(x) - 2 + c(x))i$ for $i = 1, \dots, m(x)$.*

(d) *In (b) and (c), equality holds if $i = m(x)$. Equality also holds if $i < n$ and $s_i(M, S, x) < s_{i+1}(M, S, x)$.*

Proof. We may assume $K = \mathbb{C}$, so that κ_K is algebraically closed. If $x \notin \Gamma_S$, by rescaling we may reduce the claims to an instance of Proposition 3.6.3(c), so we may assume hereafter that $x \in \Gamma_S$.

Suppose first that X is contained in the affine line; in this case, we may follow the proof of [25, Theorem 11.3.2(c)]. Namely, using Frobenius pushforwards as in Definition 3.5.2 (and using both Proposition 2.3.5 and Proposition 3.5.5), we may reduce to the case where $s_i(M, S, x) < \omega\rho_S(x)$. In this case, the claims follow by first using Corollary 2.1.6 to choose an element of M_x which is a cyclic vector for $M_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Frac}(\mathcal{O}_{X,x})$ for the derivation $\frac{d}{dt}$, then applying Proposition 2.2.6.

We now treat the case of general X. Let C be a smooth projective connected curve over κ_K with function field $\kappa_{\mathcal{H}(x)}$. Choose a nonconstant $\overline{f} \in \kappa_{\mathcal{H}(x)}$ of degree $d > 0$, then choose $f \in \mathcal{O}_{X,x}$ with $x(f) = 1$ lifting $\overline{f}$ which is unramified at each point corresponding to a branch named in (a). Note that removing part of X contained in a branch adds i to both sides of the desired inequality and is thus harmless; we can thus ensure that f defines a finite étale map $X \to X'$ for X' a subspace of the affine line. Put $x' = f(x')$ and let S' be the image of S. For each branch U' of X' at x, the slope of $\sum_{j=1}^{di} \log s_j(f_*M, S', U')$ can be computed as follows. Let P' be the point of C corresponding to U'. For each point $P \in \overline{f}^{-1}(P')$ with multiplicity m and ramification number e (so that $e = m$ if the ramification at P is tame), let U be the corresponding branch of X at x; we then get a contribution of $1 - e$ plus the slope of $\sum_{j=1}^{i} \log s_j(M, S, U)$. We thus deduce the claim from the previous case plus the Riemann-Hurwitz formula. $\square$

To show that the functions $s_i(M, S, \cdot)$ can be computed using some triangulation, we use the following criterion.

Lemma 5.3.7. *Let T be a triangulation containing S with the following properties.*

(a) *The set Γ_T meets every connected component of $X \setminus \Gamma_S$. In particular, the retraction π_T exists (see Remark 5.2.5).*
(b) *Along each edge of Γ_T, the functions $\log s_i(M, S, \cdot)$ are affine for $i = 1, \dots, n$.*
(c) *For each $x \in T$, for $i = 1, \dots, m(x)$, the slope of $\log s_i(M, S, \cdot)$ along any nonskeletal branch of X at x is 0.*

Then for $i = 1, \dots, n$, $\log s_i(M, S, \cdot)$ factors as the retraction π_T followed by a piecewise affine function on Γ_T.

Proof. Note that (b) implies that (c) holds also for $x \in \Gamma_T$ by Proposition 3.6.3(c,d). We may thus deduce the claim using Lemma 4.3.11 (applied after enlarging K to turn a virtual open disc into a true open disc) and Lemma 5.3.5. $\square$

We then obtain the following generalization of Theorem 4.5.15, which recovers the main results of [7, 35, 36].

Theorem 5.3.8. *There exists a triangulation T containing S, which is strict if S is, such that Γ_T meets every connected component of $X \setminus \Gamma_S$ (so the retraction π_T exists by Remark 5.2.5) and each function $\log s_i(M, S, \cdot)$ factors as π_T followed by a piecewise affine function on Γ_T. In particular, the functions $s_1(M, S, x), \dots, s_n(M, S, x)$ on X are continuous.*

Proof. Since X is locally compact, it suffices to check the claim locally around some $x \in X$. If $x \notin \Gamma_S$, the claim follows from Theorem 4.5.15, so we need only consider $x \in \Gamma_S$. By Theorem 5.3.6(a), we can choose T so that for $i = 1, \dots, m(x)$, the slope of $\log s_i(M, S, \cdot)$ is 0 along each T-nonskeletal branch of X at x. By Proposition 3.6.3(a), we may draw an open star in Γ_T around x such that on each edge, the functions $\log s_i(M, S, \cdot)$ are affine for $i = 1, \dots, n$. On this star, the conditions of Lemma 5.3.7 are satisfied, so the desired result follows. $\square$

One can also change the functions to match the new triangulation without disturbing the conclusion. This gives a conclusion more closely matching that of [7].

Definition 5.3.9. We say that a triangulation T is *controlling* for M if the functions $s_i(M, T, \cdot)$ also factor as the retraction π_T followed by some piecewise affine functions on Γ_T. That is, we must be able to take $T = S$ in the conclusion of Theorem 5.3.8.

Corollary 5.3.10. *In the notation of Theorem 5.3.8, the triangulation T is controlling.*

Proof. This follows from Theorem 5.3.8 and the fact that conditions (a,b) of Lemma 5.3.7 can be stated in terms of intrinsic subsidiary radii, and so remain valid if we replace S by T. $\square$

Remark 5.3.11. The methods of [7] and [35, 36], when considered without reference to this paper, can only prove a weaker version of Theorem 5.3.8: they only provide a controlling triangulation over a sufficiently large analytic field K' containing K. Again the problem is that this triangulation may involve vertices which project to type 4 points of the original curve, which our methods are able to rule

out. In the language of [5, 7], we are able to exhibit a controlling strictly semistable model already over $\mathbb{C}$, whereas the methods of [7] and [35, 36] provide such a model only a possibly larger algebraically closed analytic field containing $\mathbb{C}$.

5.4. **Clean decompositions.** One has an analogue of the spectral decomposition for the stalk of M at a point $x \in X$. Using Theorem 5.3.8, we can extend this decomposition to specific subspaces of X.

Lemma 5.4.1. *Choose $x \in X$ of type 2 or 3.*

(a) *There exists a unique direct sum decomposition $M_x = \bigoplus_i N_i$ whose base extension to $\mathcal{H}(x)$ is the spectral decomposition.*
(b) *There exists a finite étale extension S of $\mathcal{O}_{X,x}$ such that $M_x \otimes_{\mathcal{O}_{X,x}} S$ admits a direct sum decomposition whose base extension to $\mathcal{H}(x) \otimes_{\mathcal{O}_{X,x}} S$ is a refined decomposition.*

Proof. Part (a) follows by using the pushforward argument from the proof of Theorem 5.3.6 to reduce to the case where X is contained in the affine line over K, then following the proof of [25, Theorem 12.3.2]. Part (b) follows similarly upon noting that the local ring $\mathcal{O}_{X,x}$ is henselian. □

Theorem 5.4.2. *Let T be a controlling triangulation for M.*

(a) *For $x \notin \Gamma_T$, let U be the branch of $\pi_T(x)$ containing x. Then the restriction of M to U splits as a direct sum in which for each summand N, there exists a constant $c > 0$ such that $s_i(N, T, y) = c$ for $i = 1, \dots, \operatorname{rank}(N)$ and $y \in U$.*
(b) *For $x \in \Gamma_T$, let E be the open star around x (i.e., the union of x with the interiors of the edges of Γ_T incident upon x) and put $U = \pi_T^{-1}(E)$. Then the restriction of M to U splits as a direct sum in which for each summand N, the functions $s_i(N, T, \cdot)$ on U for $i = 1, \dots, \operatorname{rank}(N)$ are all equal and admit $T \cap U$ as a controlling triangulation.*

Proof. Part (a) is immediate from Proposition 3.6.7. To obtain (b), first apply Lemma 5.4.1 to obtain a decomposition over an uncontrolled open neighborhood V of x. Note that V already contains all but finitely many branches of X at x. To fill in the remaining branches, apply (a) (for T-nonskeletal branches at x) and Proposition 3.6.9 (for T-skeletal branches at x). □

Remark 5.4.3. The decompositions appearing in Theorem 5.4.2 are analogues of the *good formal structures* for formal meromorphic connections described in [26, 27]. Additional analogues in the p-adic setting also appear in [32]. The decompositions given here can be used

to obtain a global index formula for connections on analytic curves, in the style of the work of Robba [37, 38, 39, 40] and Christol and Mebkhout [11, 12, 13, 14]. Such a formula will appear in upcoming work of Baldassarri.

Remark 5.4.4. Using these results, it is tempting to look for a more global version of Theorem 3.8.17. When $p > 0$, one might even guess that every connection étale-locally satisfies the Robba condition. However, this guess is incorrect as shown by Remark 2.3.18, and it is not immediately obvious to us how to salvage the statement.

One motivation for doing so would be to show that the behavior of radii of convergence for connections arising from discrete representations of the geometric fundamental group, which can be explained in terms of Faber's Berkovich-theoretic ramification locus [18, 19], is in fact completely representative of the general case.

References

[1] Y. André, Filtrations de type Hasse-Arf et monodromie p-adique, *Invent. Math.* **148** (2002), 285–317.

[2] Y. André, Slope filtrations, *Confluente Math.* **1** (2009), 1–85.

[3] M. Baker and R. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line*, AMS Surveys and Monographs 159, Amer. Math. Soc., Providence, 2010.

[4] M. Baker, S. Payne, and J. Rabinoff, Nonarchimedean geometry, tropicalization, and metrics on curves, arXiv:1104.0320v2 (2012).

[5] F. Baldassarri, Continuity of the radius of convergence of differential equations on p-adic analytic curves, *Invent. Math.* **182** (2010), 513–584.

[6] F. Baldassarri and L. Di Vizio, Continuity of the radius of convergence of p-adic differential equations on Berkovich spaces, arXiv:07092008v1 (2007).

[7] F. Baldassarri and K.S. Kedlaya, On the controlling graph of connections on non-archimedean curves, preprint (in preparation).

[8] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.

[9] V.G. Berkovich, Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. IHÉS* **78** (1993), 5–161.

[10] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis*, Grundlehren der Math. Wiss. 261, Springer-Verlag, Berlin, 1984.

[11] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des quations différentielles, *Annales Inst. Fourier* **43** (1993), 1545–1574.

[12] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des quations différentielles, II, *Annals of Math.* **146** (1997), 345–410.

[13] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des quations différentielles, III, *Annals of Math.* **151** (2000), 385–457.

[14] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des quations différentielles, IV, *Invent. Math.* **143** (2001), 629–672.

[15] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, reprint of the 1962 original, Amer. Math. Soc., Providence, 2006.

[16] A.J. de Jong, Étale fundamental groups of non-Archimedean analytic spaces, *Compos. Math.* **97** (1995), 89–118.

[17] A. Ducros, La structure des courbes analytiques, preprint (in preparation).

[18] X. Faber, Topology and geometry of the Berkovich ramification locus for rational functions, arXiv:1102.1432v3 (2011).

[19] X. Faber, Topology and geometry of the Berkovich ramification locus for rational functions, II, arXiv:1104.0943v2 (2011).

[20] K. S. Kedlaya, A p-adic local monodromy theorem, *Annals of Math.* **160** (2004), 93–184.

[21] K. S. Kedlaya, Local monodromy of p-adic differential equations: an overview, *Int. J. Number Theory* **1** (2005), 109–154; errata at `http://math.ucsd.edu/~kedlaya/papers/`.

[22] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212.

[23] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672.

[24] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at monomial valuations, *Compos. Math.* **145** (2009), 143–172.

[25] K.S. Kedlaya, *p-adic Differential Equations*, Cambridge Univ. Press, Cambridge, 2010; errata posted at `http://math.ucsd.edu/~kedlaya/papers`.

[26] K.S. Kedlaya, Good formal structures for flat meromorphic connections, I: Surfaces, *Duke Math. J.* **154** (2010), 343–418; erratum, *ibid.* **161** (2012), 733–734.

[27] K.S. Kedlaya, Good formal structures for flat meromorphic connections, II: Excellent schemes, *J. Amer. Math. Soc.* **24** (2011), 183–229.

[28] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, IV: Local semistable reduction at nonmonomial valuations, *Compositio Math.* **147** (2011), 467–523.

[29] K.S. Kedlaya, Nonarchimedean geometry of Witt vectors, arXiv:1004.0466v5 (2012); to appear in *Nagoya Math. J.*

[30] K.S. Kedlaya and R. Liu, Relative p-adic Hodge theory, I: Foundations, arXiv:1301.0792v1 (2013).

[31] K.S. Kedlaya, J. Pottharst, and L. Xiao, Cohomology of arithmetic families of (φ, Γ)-modules, arXiv:1203.5718v1 (2012).

[32] K.S. Kedlaya and L. Xiao, Differential modules on p-adic polyannuli, *J. Institut Math. Jussieu* **9** (2010), 155–201.

[33] Z. Mebkhout, Analogue p-adique du théorème de Turrittin et le théorème de la monodromie p-adique, *Invent. Math.* **148** (2002), 319–351.

[34] S. Payne, Analytification is the limit of all tropicalizations, *Math. Res. Lett.* **16** (2009), 543–556.

[35] J. Poineau and A. Pulita, The convergence Newton polygon of a p-adic differential equation II: Continuity and finiteness on Berkovich curves, arXiv:1209.3663v1 (2012).

[36] A. Pulita, The convergence Newton polygon of a p-adic differential equation I: Affinoid domains of the Berkovich affine line, arXiv:1208.5850v2 (2012).

[37] P. Robba, On the index of p-adic differential operators, I, *Annals of Math.* **101** (1975), 280–316.

[38] P. Robba, On the index of p-adic differential operators, II, *Duke Math. J.* **43** (1976), 19–31.

[39] P. Robba, On the index of p-adic differential operators, III, Application to twisted exponential sums, *Astérisque* **119–120** (1984), 191–266.

[40] P. Robba, Indice d'un operateur differentiel p-adique, IV, Cas de systèmes. Mesure de l'irrégularité dans un disque, *Annales Inst. Fourier* **35** (1985), 13–55.

[41] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math. 265, Springer-Verlag, Berlin, 1972.

[42] P. Schneider and J. Teitelbaum, Algebras of p-adic distributions and admissible representation. *Invent. Math.* **153** (2003), no. 1, 145–196.

[43] J.-P. Serre, *Local Fields*, Graduate Texts in Math. 67, Springer-Verlag, New York, 1979.

[44] M. Temkin, Stable modification of relative curves, *J. Alg. Geom.* **19** (2010), 603–677.

[45] M. Vaquié, Valuations, in *Resolution of singularities, Obergurgl, 1997*, Progress in Math. 181, Birkhäuser, Basel, 2000, 539–590.

[46] L. Xiao, Non-archimedean differential modules and ramification theory, thesis, Massachusetts Institute of Technology, 2009.

[47] L. Xiao, On refined ramification filtrations in the equal characteristic case, arXiv:0911.1802v2 (2011), to appear in *Alg. and Num. Theory*.